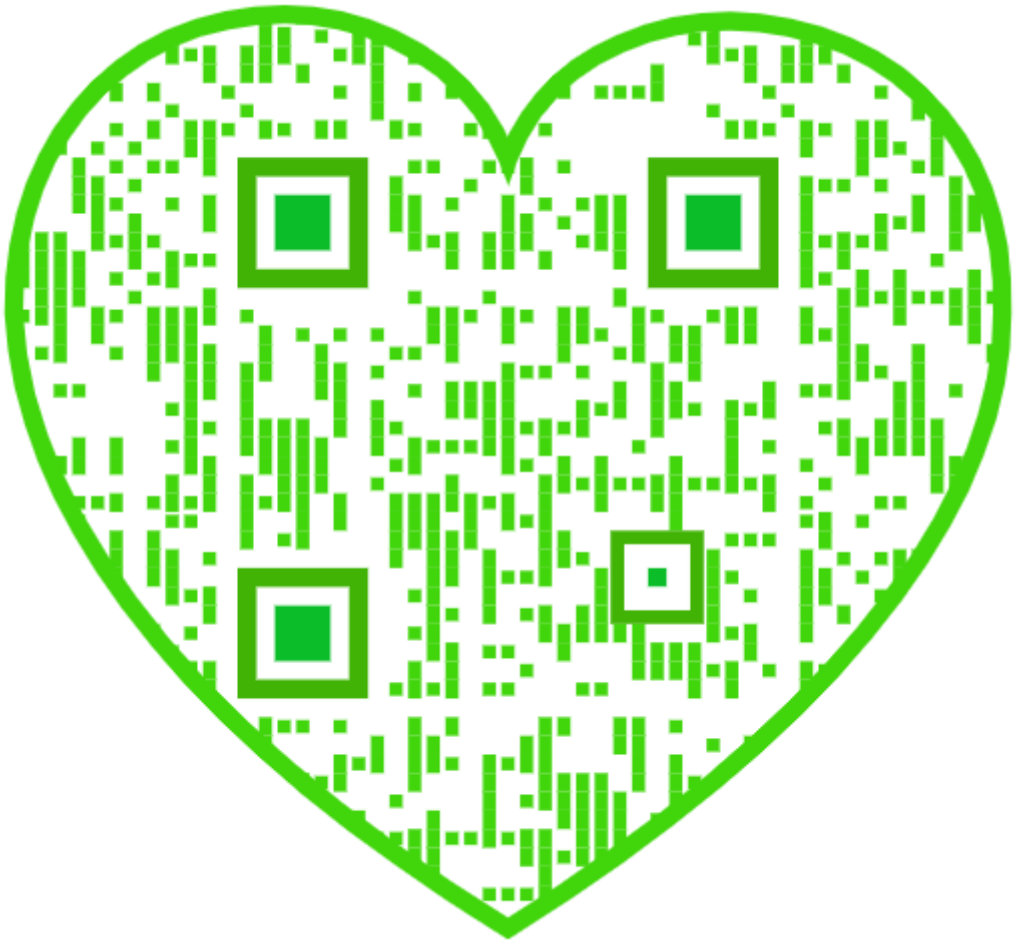

VELOCITY DIAGRAMS.

THEIR CONSTRUCTION AND THEIR USES.

INTENDED FOR ALL WHO ARE INTERESTED IN MECHANICAL MOVEMENTS.

BY

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PREFACE.

THIS treatise is in effect an abstract of a series of lectures forming a part of the course of instruction at the Stevens Institute of Technology. It explains the principles of the more common and convenient graphic processes of determining at any given instant the direction and velocity of the motion of a point, whether that motion be constant or variable.

It is intended not only for use in the classroom, but for the benefit of those who may wish to study the subject without the aid of an instructor.

C. W. MACCORD.

HOBOKEN, N. J., Oct. 12, 1901.

VELOCITY DIAGRAMS.

1. It is a familiar fact that in the operation of any piece of mechanism, the parts go through a series of motions in regular order, finally returning to their original positions; after which the same series of motions is repeated, and so on indefinitely. One complete series is called a **cycle**; in completing which it frequently happens that, supposing the first or driving piece to move uniformly in one direction, the motions of other parts will vary either in velocity or direction, and often in both. And in studying the action of any mechanical movement, or in comparing the actions of different ones, it is often desirable to have a clear understanding of the law of variation, in regard to the motion of a given piece or a given point.

Now, assuming that for a given motion of the driver, the motion of the point considered can be determined at any instant, or in other words in any phase of the action—then it is beyond question that a graphic representation is the best if not indeed the only means of conveying to the mind a distinct and comprehensive idea of the law according to which the motion varies in velocity and direction.

Such a representation, or "**velocity diagram**," is shown in Fig. 1. It consists merely of a curve whose abscissas, set

off from left to right upon the line *MN*, represent times, and the ordinates 11′, 22′, etc., represent the velocities of the moving point at the instants indicated by the points 1, 2, etc.; the positive ordinates, or those above the line, indicate motion in one direction, that in the opposite direction being indicated by the negative ordinates, below the line. And a single glance at this figure is sufficient to establish the claim above made,—it gives in an instant all the information that could be gathered from lengthy explanations and tables of figures.

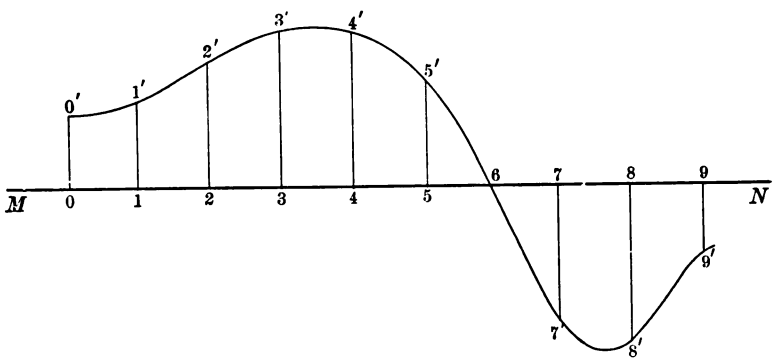


FIG. 1.

2. Now, given the velocity of the driver, how to determine the values of these ordinates? We have here to choose between two methods of procedure, the graphic and the analytic. There is no question that the members of a train of mechanism can be represented by symbols, the laws of their motions embodied in formulæ, and the desired values ascertained by algebraic computation. In the graphic method, the motion of a point at any instant is represented in magnitude and direction by a right line of definite length; and relations may be established between lines thus representing the motions of properly selected points, and other

lines closely connected with the moving pieces, such that the values sought can be determined by geometric reasoning. Of the two, the latter method is preferable for ordinary use, being far more simple and expeditious than the former, while the accuracy attainable is quite sufficient for practical purposes. Its foundations lie upon a few kinematic principles, of which we will briefly state the most important.

3. The first is the **composition** of motion. Suppose the point A, Fig. 2, to receive at the same instant two impulses, which separately would impart to it the motions represented in direction and velocity by the lines AB, AC; these are called **components**, and the **resultant** of these, which is the actual direction and velocity of the motion of the point, is

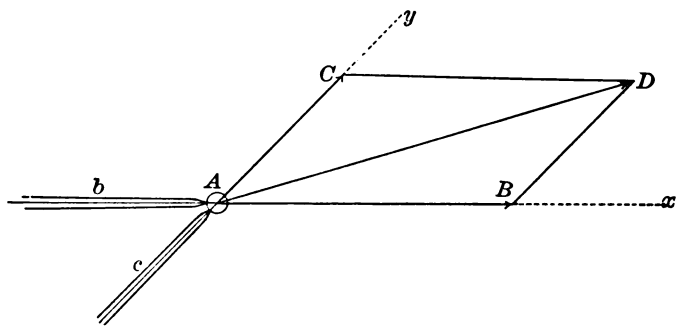


FIG 2.

AD, the diagonal of the parallelogram $ABCD$. The condition of things here is that A is a free point in space,—as if, by way of illustration, it were a billiard-ball impelled by the simultaneous strokes of two cues b and c, which are not connected with each other or with the ball; a consideration which, as will subsequently appear, is of no small importance. There may be more than two components; in that case, find the resultant of any two of them, compound that with any one of the others, and so on to the end. If there be three

components not in the same plane, these will be three adjacent edges of a parallelopipedon, and the resultant will be the body diagonal which passes through the moving point: our attention, however, will be chiefly confined to motions in one plane or, what is practically the same thing, in parallel planes.

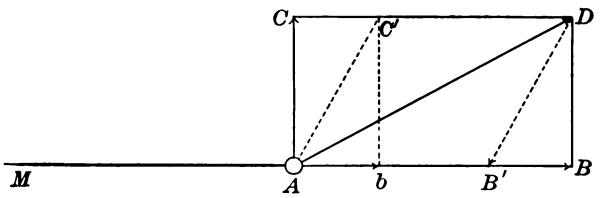
4. The **resolution** of motion is the exact converse of the preceding. If a motion can be determined by compounding two others, that motion, if given, can be separated or *resolved* into its original components. Thus, in Fig. 2, suppose the motion AD to be assigned, and let it be required to determine two components having the directions Ax, Ay, of which AD shall be the resultant. By drawing through D lines parallel to Ax and Ay, it is evident that we shall limit the required components, AC, AB. But AD may be the diagonal of any one of an infinite number of parallelograms; whence it follows that a given motion may be resolved into two components respectively parallel to any two lines having different directions.

5. Side Component and **Longitudinal Component.**—In Fig. 3, let the point A, of the right line MA, have a motion represented by AD. Resolve this into the components AC perpendicular to MA, and AB in the direction of that line; then AC is called the side component, and AB the longitudinal component. And these components always exist, no matter how the motion AD may be resolved. Thus, if it be resolved into AC', AB', then AC' itself has a component Ab along MA, and Ab is equal to $B'B$, so that the total longitudinal component is $AB' + Ab, = AB$.

Again, in Fig. 4, let AD be resolved into AE, AF; then AE can be resolved into the rectangular components Ae, Ag, and AF into the pair Af, Ah; also $Af = eB$, and

$Ah = Cg$; Ae and Af lie in the same direction, and the total longitudinal component is $Af + Ae, = AB$; but since Ag and Ah lie in opposite directions, the actual side component

FIG. 3.



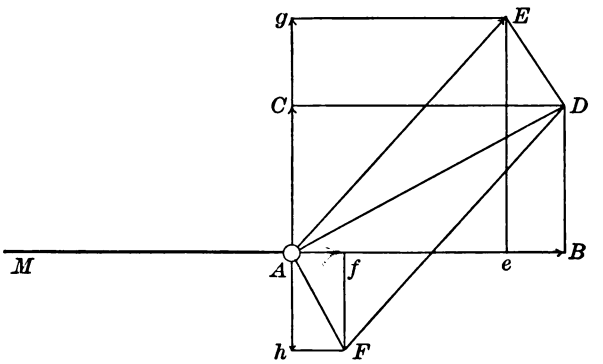


FIG. 4.

is $Ag - Ah, = AC$ as before. These rectangular components AB, AC, may properly be designated *absolute components.*

6. In Fig. 5, let AM represent a rigid and inextensible line—as a piece of stiff steel wire, and let the point A have a motion AD, of which AF and AE are the side and longitudinal components respectively. Then any other point, B, of this wire must of necessity have a longitudinal component BG, equal to AE, and in the same direction. And whatever the actual motion of the point, it follows from the preceding that the other component must be perpendicular to AM, so that the extremity of the resultant must lie in the

indefinite vertical line xx drawn through G. If, then, the direction By is assigned, the intersection H of xx and By will determine BH.

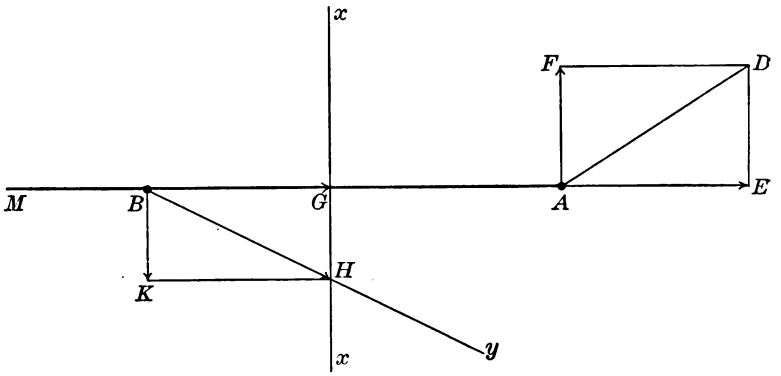


FIG. 5.

7. In Fig. 6, AM, AN, represent two rigid bars, pivoted together at A; let AE, AF, be the absolute longitudinal components, from which it is required to find the motion of

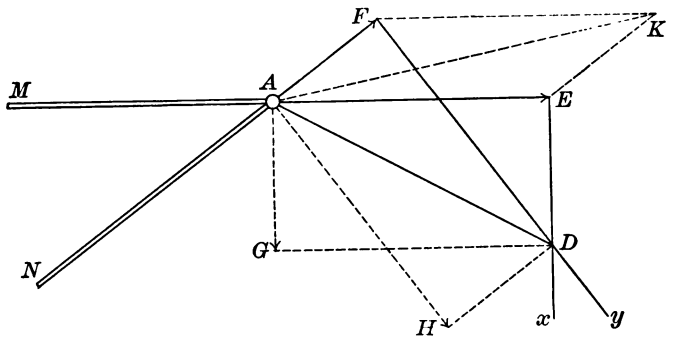


FIG. 6.

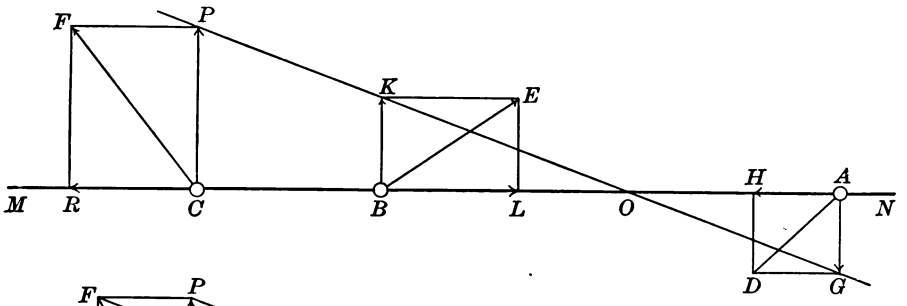
A. From what has just been shown, the extremity of the resultant must lie in Ex perpendicular to AM, and also in Fy perpendicular to AN; it must, then, be their point of intersection D. Which is obviously as it should be, since the

resultant AD thus determined can be resolved either into the rectangular components AE, AG, or into the pair AF, AH. Had we proceeded as in Fig. 1, by completing the parallelogram of which AE, AF are the sides (as one not familiar with the previous reasoning would be very likely to do), the diagonal AK would have neither the right magnitude nor the right direction, unless AM and AN were perpendicular to each other: and, as will be seen subsequently, there are other cases than this in which that procedure will give erroneous results.

8. Effect of Side Components.—In Fig. 7, let MN be an inflexible steel rod, upon which the perforated balls A, B, C, are free to slide; then the centres of these balls are points *upon* the right line MN, but not *of* it, and, whatever their motions may be, the longitudinal components AH, BL, CR do not affect the line, nor do they have any relation to each other, and they need not have either the same magnitude or the same direction. But obviously the side components do affect the line,—if any two of them are equal, and lie on the same side of MN, the whole line must be translated bodily in the direction of those two, and the side components of all its points must be equal. But if motions be assigned to any two of these balls, such that their side components either lie upon opposite sides of MN or are unequal if they lie on the same side, then these two side components will establish a rotation of the line about some point of the line itself. If, for example, we give to A and B the motions AD and BE, their side components are AG and BK respectively, and, drawing GK, it is perceived that MN must turn about the point O where GK cuts it. Moreover, this latter line determines the directions and the values of the side components of the motions of all other points on MN; thus any point C,

to the left of O, must move upward, and that at a rate CP limited by the prolongation of GK; any point to the right of O must move downward in like manner, while if a ball were placed at O, it could not move sidewise at all, though it

FIG. 7.



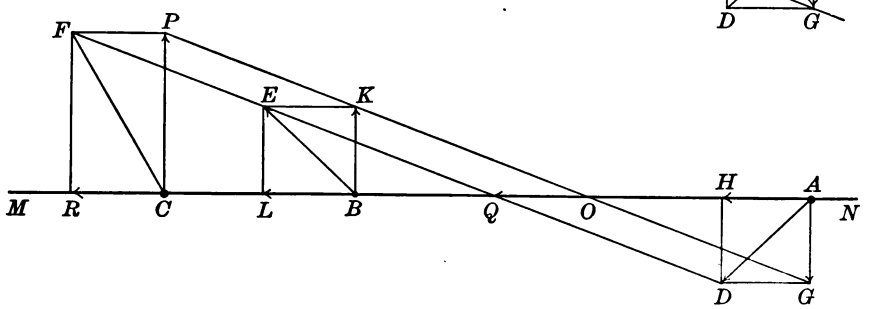


FIG. 8.

might move endwise in either direction and with any velocity.

9. Now if, as in Fig. 8, we consider A, B, C, to be points *of* the inflexible and inextensible line MN, and therefore as remaining always at fixed distances from each other; it is still true that the side components AG, BK, of the simultaneous motions of two points, A and B, for example, determine the rotation of MN about O. But supposing AD to be assigned, the motion of B, for instance, would no longer be entirely arbitrary, since the longitudinal component BL

must be equal to AH and in the same direction. This is also true of every point of MN, including O, whose absolute motion must therefore be OQ, $= AH$. Consequently in this case, since GD, OQ, KE, PF, are all equal, and all parallel to MN, the line $DQEF$, joining the extremities of the resultants, will be equal and parallel to $GOKP$, joining the extremities of the side components.

10. Representation of Angular Velocity.—The linear velocity of a point at unit distance from a centre about which the point travels in a circular arc, is the measure of the *angular velocity* of the point about that centre. From which it follows, that whatever the distance of the point from the centre, we shall always have the value, angular velocity $= \frac{\text{linear velocity}}{\text{radius}}$.

If then, as in Fig. 9 the point A, revolving about the

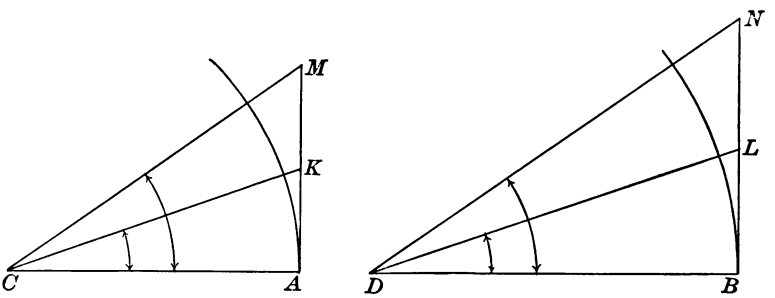


FIG. 9.

centre C, has the linear velocity AK, the angle ACK represents the angular velocity; if it be desired to find the linear velocity of B in rotating with the same angular velocity about D, we have only to make the angle BDL equal to ACK. If the angular velocity of B is to be made twice that of A, however, it is to be noted that this is not done by

doubling this angle; but we make $AM = 2AK$, and then make the angle BDN = angle ACM.

11. The Instantaneous Axis of Rotation.—In Fig. 10, let AB represent an inflexible and inextensible rod, and first let the points A and B move in the plane of the paper, in opposite directions perpendicular to AB, with velocities AF, BK: these motions establish a rotation of the rod about the point I, determined by drawing FK. If we now add a longi-

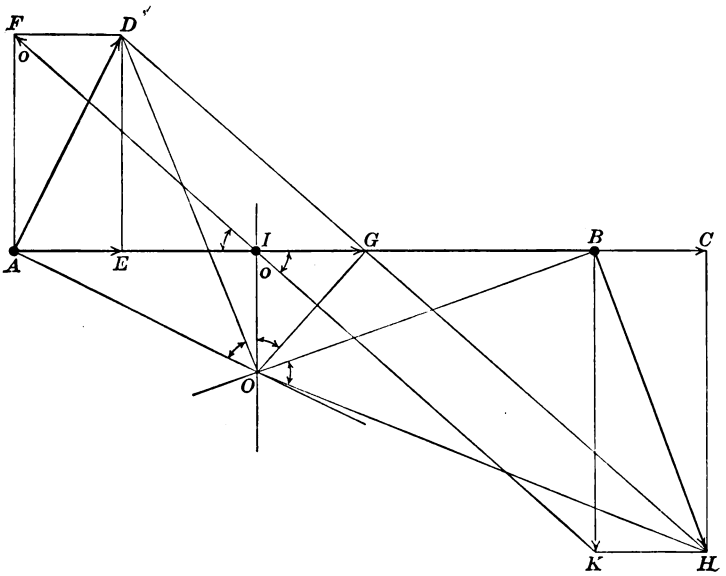


FIG. 10.

tudinal component to the motion of each point, as AE, = IG, = BC, the resultant motions of A and B will become AD and BH, and that of I will be IG.

Draw, through A and I, lines perpendicular to AD and IG respectively, intersecting in O; draw also DO and GO. Then the triangles AFD, AIO, are similar, whence

$$\frac{AF}{AI}, = \frac{AD}{AO}, = \frac{FD}{OI}, = \frac{IG}{IO}.$$

Therefore the angles AOD, GOI, are equal; that is to say, the two points A and I are rotating with the same angular velocity and in the same direction about the centre O. The same must therefore be true of every other point of the rigid bar. This may be at once verified as to the point B, as follows; we have already shown that $\frac{AD}{AO} = \frac{AF}{AI}$; therefore the angles AOD, AIF, BIK, are equal, whence

$$\frac{AF}{AI}, = \frac{BK}{BI}, = \frac{IG}{OI} = \frac{KH}{OI}.$$

Therefore the triangles BKH, BIO, are similar, and BH is perpendicular to BO.

With reference to the bar AB, the point O is called the **instantaneous axis,** because during the motion of the bar, it changes its position from instant to instant, not only in space but relatively to the bar itself; and it is found, when the simultaneous motions of two points of the bar are given, as for example AD, BH, by drawing through each point a line perpendicular to the direction of its motion; the intersection of these perpendiculars locates the axis sought.

And obviously, since the bar is rigid, any point rigidly connected with it must obey the same law, and rotate about the instantaneous axis with the same angular velocity and in the same direction.

12. The Instantaneous Centre.—The rotation about O, then, is equivalent to a rotation about I, in the same direction and with the same angular velocity, combined with a longitudinal motion.

As a distinctive name, then, this point I may properly be called the **instantaneous centre.**

It is, as we have seen, the foot of the perpendicular let fall from O upon AB. This fact is sometimes of service in locating the instantaneous axis—as, for instance, when the directions of the motions of A and B being assigned, the perpendiculars to those directions, Ax and By, intersect each other very acutely.

Now, in Fig. 10, FK cuts OG in L; and in the triangles AIF, IOL, the angles AFI, OIL, are equal by reason of the parallels AF, OI; and the angles AIF, IOL, have already been shown to be equal. Therefore, since IA is perpendicular to AF, IL is perpendicular to OL; consequently OG is perpendicular to DH, which passes through G, and is parallel to FK, as was proved by Fig. 8.

That is to say, G is the foot of the perpendicular let fall from the instantaneous axis upon the line joining the extremities of the resultants.

13. Contact Motions—Normal and Tangential Components.—In Fig. 11 are shown two pieces turning about the fixed cen-

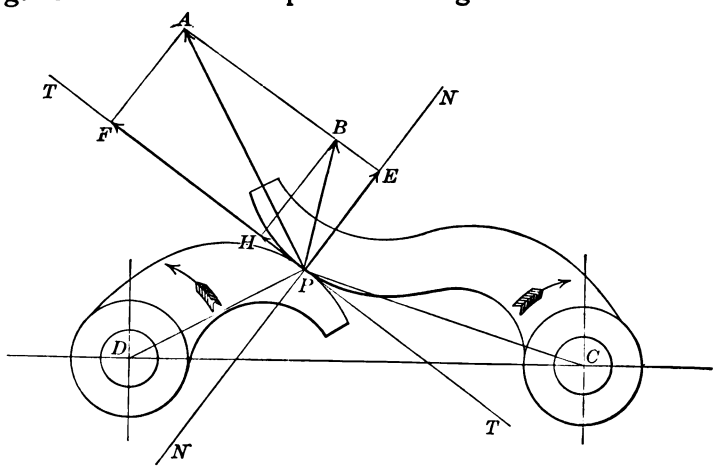


FIG. 11.

tres C and D, and in contact at P; if the left-hand one turn as shown by the arrow, it will push the other out of its

way, and compel it to turn, in this case in the opposite direction. Draw the contact radii *PC* and *PD*; also draw through *P*, *TT* the common tangent to the two curves, and *NN* their common normal. The point *P* of the driver must move in a direction perpendicular to *PD*; let its linear velocity be represented by *PA*, which can be resolved into the normal component *PE* and the tangential component *PF*. Of these two, the latter is obviously non-effective; it represents merely the sliding of *P* along the tangent, and has no tendency to move the follower. The motion of the point *P* of the right-hand piece must be perpendicular to *PC*, and must have a velocity such that its normal component shall also be *PE*;—consequently the extremity *B* of this resultant must lie in the line *AE*. The component *PH* also represents sliding along the tangent; it is clear, therefore, that in this case the actual sliding of one piece upon the other will be *PF* — *PH*, or *HF*.

14. Roberval's Method of Drawing Tangents.—Whatever the path described by a moving point, the direction of its motion at any instant is that of the tangent to that path, at the position occupied by the point at the given instant.

The law of the motion, even in abstract mathematics, may often be best explained by describing it as dependent upon, and produced by, other motions: thus, for example, a spiral of any kind is usually defined as the path of a point which travels along a right line, while the line itself revolves about a fixed centre.

And "**Roberval's method**" of drawing a tangent to a curve consists simply in finding the resultant of these component motions. A simple illustration of this method is shown in its application to the spiral of Archimedes, Fig. 12. Let a point move at a uniform rate from *P* along the right line *PQ*,

while that line revolves in the direction of the arrow around *P* as a fixed centre, also at a uniform rate. Let the radial travel be such that, while making one revolution, the point shall move from *P* to *A*, then it will trace the curve *PEFAG*; and let it be required to draw a tangent at the point *O*. The two motions might be supposed to take place independently; if we first imagine the rotation to be arrested, the

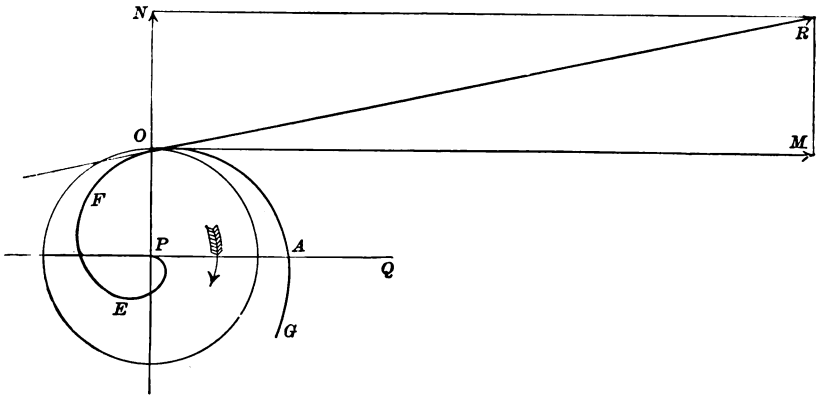


FIG. 12

point will move, in the time of one revolution, radially outward through a distance *ON* equal to *PA*: and we may therefore let *ON* represent the velocity of this component. If we suppose the radial motion to be arrested, then in the same time the point will describe the circumference of the circle of which *PO* is the radius. The direction of this motion at the instant being that of the tangent to the circular path, we have as the other component *OM* perpendicular to *PO*, and equal in length to the circumference; then, completing the rectangle, the diagonal *OR* is tangent at *O* to the spiral, as required: obviously, both components may be reduced in the same proportion, without affecting the direction of the resultant, which is of course independent of the actual velocity of the tracing point.

Since $\frac{NR}{ON}$ = tan. *NOR*, the inclination of the tangent to the radius vector can be determined by elementary trigonometry; and in its application to many other curves, of high and low degree, a like simplicity characterizes this elegant process. Nevertheless, in his treatise on Descriptive Geometry, Mr. J. F. Heather makes this curious remark: "This method, which Roberval invented before Descartes had applied algebra to geometry, is implicitly comprehended in the processes of the differential calculus, on which account it is not noticed in elementary mathematics";— where it would seem, on the contrary, to deserve a conspicuous place: it is certainly more easily comprehended than the calculus, to which indeed it is a natural prelude. At any rate, it has a natural and direct application to our present purpose, since in mechanical devices the actual motion of a point is, more often than not, controlled by other motions whose combined effect it is necessary to determine.

15. But in applying it, all the circumstances of the case must be considered, and all the conditions which may affect the result must be satisfied. This suggestion may at first glance appear superfluous; but it seems otherwise in view of the fact that Mr. Heather, in the only illustration of Roberval's method which he gives, has conspicuously failed to act upon it. The curve selected for this solitary example is the ellipse, traced, as in Fig. 13, by a point *P* moving in such a manner as to keep always taut a thread *APB* attached by its extremities to the foci *A* and *B*.

Mr. Heather's explanation is as follows: "Since the length of the string is constant, the distance *AP* is lengthened at each instant of the motion by the same distance as the distance *BP* is diminished. The velocity of the describing

point in the direction AP is therefore equal to the velocity in the direction PB. If, then, equal straight lines be cut off from PB, and from AP produced, and the parallelogram $PNOM$ be completed, the diagonal PO of this parallelogram will be the direction of the motion of the generating point at P, and consequently the tangent to the curve at this

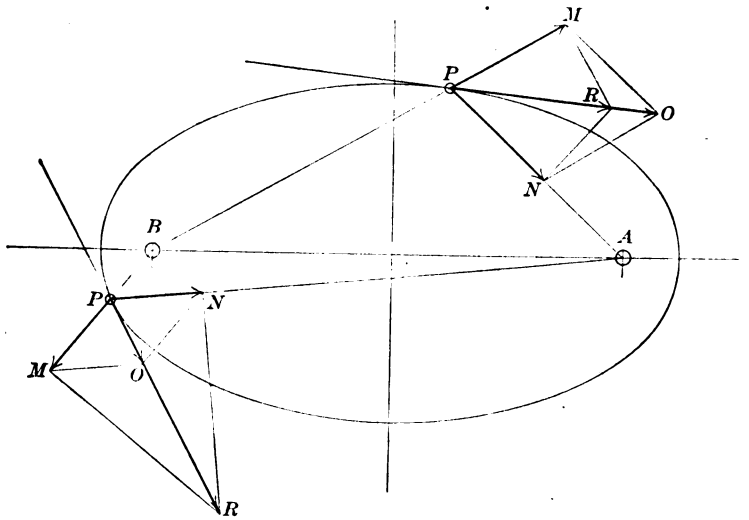


FIG. 13.

point. It is clearly seen from this, that in the ellipse, the tangent bisects the angle BPN formed by one of the focal distances and the production of the other," etc., etc.

Mr. Heather elsewhere explicitly states that if the components in two directions are given, whatever their relative magnitudes, the method of Roberval consists in "completing the parallelogram and drawing the diagonal." In the above example the *direction* of the tangent is, undeniably, correctly found in this manner—and the same is true if this construction is applied to the hyperbola, in which the moving point recedes from both foci at the same rate, or to the

parabola, in which it recedes at the same rate from the focus and the directrix.

16. In all these cases, however, the components are *equal;* but the statement above quoted leads us to expect a correct determination whether they are equal or not: let us put it to the test. In Fig. 14, let A and B be two fixed points, with reference to which the point P moves subject to the condition that PB shall always be twice PA. In other words, the path of P is the locus of the vertices of all triangles of which AB is the common base, and one of the two other sides is twice the third. In order to preserve this ratio it is clear that P, in moving to the right, must recede

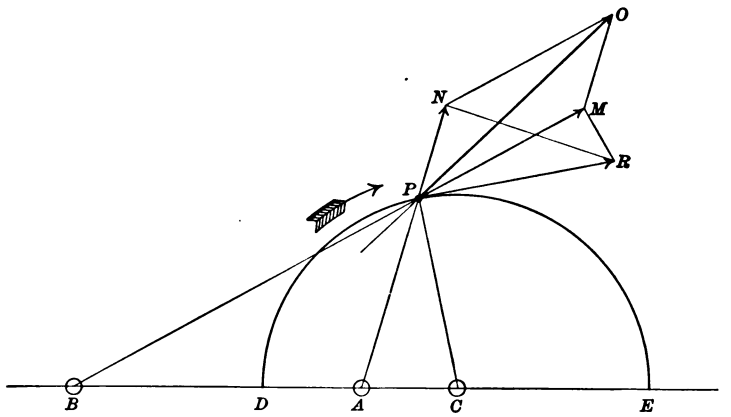


FIG. 14.

from B twice as fast as it does from A. Consequently, setting off on the prolongations of BP and AP, $PM = 2PN$, and completing the parallelogram, the diagonal PO should, in accordance with what immediately precedes, be the required tangent. But it can be shown that the path in question is the circumference of the circle whose diameter is DE, determined by making $BD = 2AD$, and $BE = 2AE$. It is

obvious that *PO* is *not* tangent to this circle, and therefore cannot be the resultant motion of *P*.

A little reflection will show that two important conditions have been neglected; the point *P* lies upon the right line *PA*, and must always do so; therefore, if that point moves as shown, the line must turn about *A* as a fixed centre; and similar reasoning applies to the line *PB*. Thus the situation is in fact more complicated than it seemed at first, and really presents a special case of the general problem of determining the motion of the intersection of two right lines rotating about fixed centres; which therefore must next receive attention. The elements of this problem are embodied in the mechanical combination shown in Fig. 15, where *AC*, *BD*, represent two steel rods, each formed into an eye at one end and turning about the fixed pins *C*, *D*; upon these rods two sleeves slide freely, and they are pivoted together by a pin at *P*, whose axis, perpendicular to the paper, intersects the centre lines of both rods.

The operation of this device may perhaps be best studied by first supposing one rod, as *AC*, to be held stationary, while *BD* turns. The point *P* of that rod must then at the instant move in a direction perpendicular to *BD*; let its velocity be represented by *PG*. The pin connecting the sleeves must move absolutely in the direction *PC*, since the rod *AC* now forms a fixed guide, along which the sleeve through which it passes is compelled to slide. The other sleeve, however, not only rotates with *BD*, but can slide along that rod: consequently the actual velocity *PI* of the pin *P* is found by drawing through *G* a parallel to *BD*. Drawing *IH* parallel to *PG*, we observe that upon the supposition above made the pin *P* moves in the direction *AC* at the rate *PI*, and in the direction *BD* at the rate *PH*.

Next let BD be held stationary, and let the point P of the rod AC move in rotation about C with the velocity PL. By similar reasoning we shall find the resultant motion of the pin P to be PF in the direction BD, and it will be accompanied by a motion PE in the direction AC.

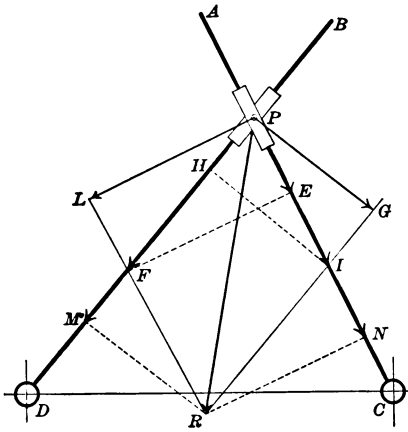


FIG. 15.

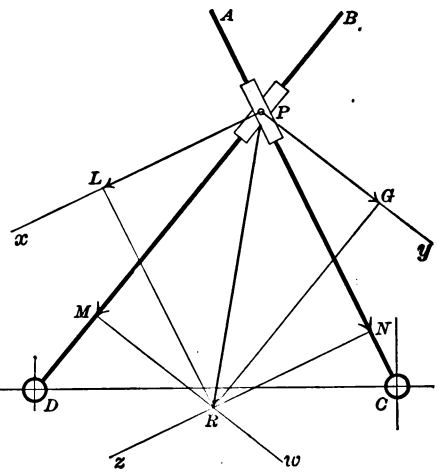


FIG. 16.

17. Now if both rods rotate at once, with the same velocities as before, the final resultant motion PR of the pin P is found by considering the partial resultants PI, PF, as components, and completing the parallelogram; because, as we have just seen, these are wholly independent of each other. But it is to be noted that P will move toward C with a velocity equal to $PI + PE$, and toward D with a velocity equal to $PF + PH$. Drawing RM perpendicular to BD, and RN perpendicular to AC, we have $FM = PH$, and $IN = PE$: so that PM, PN, are the velocities of P in the directions BD, AC, respectively.

Also, since FR, IR, are merely prolongations of LF and GI, it will be seen that having assigned the components of

rotation, *PG* and *PL*, the resultant *PR* may be at once determined by drawing perpendiculars to them, which will intersect in *R*; then drawing *RM* and *RN*, we determine *PM* and *PN*, the total sliding components.

If then, as in Fig. 16, the components *PN*, *PM*, are assigned, the resultant is found, not by completing the parallelogram, but by drawing *Nz*, *Mw*, respectively, perpendicular to *AC* and *BD*; these intersect in *R*, and *PR* is the resultant. Had this resultant been assigned, the sliding components are found at once as above stated, by reversing this process; and the components of rotation are determined as readily by drawing *Px* perpendicular to *AC*, and *Py* perpendicular to *BD*, upon which lines we let fall from *R* the perpendiculars *RL*, *RG*.

18. This, then, is the proceeding which should have been adopted in Figs. 13 and 14. Applying it in those cases, we find the resultant motion of the point *P* upon the ellipse to be *PR* instead of *PO*; the direction is the same, but the magnitude is different, and plainly will be so except in the case when *PA* and *PB* are perpendicular to each other: in Fig. 13, when the angle *APB* is obtuse, as shown at the right, *PO* is too great; and when *APB* is acute, as shown at the left, *PO* is too small. And in Fig. 14 it is seen that the direction as well as the magnitude of the true resultant *PR* is widely different from that of *PO*; moreover, it can be proved that *PR* is in this case perpendicular to the radius *PC*, and therefore tangent, as it should be, to the circular path *DPE*.

The manner of determining the resultant, when the components *PM*, *PN*, Fig. 16, are assigned, is in this case precisely the same as that employed in Fig. 6. But the conditions are by no means the same; in that instance the lines

along which the given components lie intersect always in the same point A, whereas in this the point of intersection moves along both lines; and they both clearly illustrate the fact that Roberval's method does not always consist in merely completing the parallelogram of two given components and drawing the diagonal.

19. The problem under consideration has thus far been limited by supposing the fixed centres of rotation to lie upon the lines themselves. It is clearly not essential that this should be so,—one or both these centres may be otherwise located; and the latter case is illustrated in Fig. 17, where the rigid rods AS, BT, are riveted into, and form

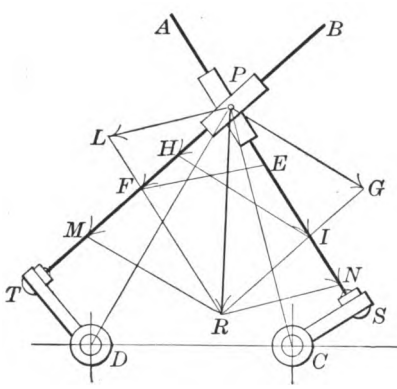


FIG. 17.

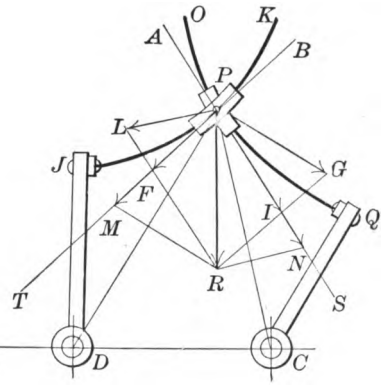


FIG. 18.

parts of, the arms CS, TD. Suppose, as before, one piece, as ASC, to be held stationary, and let the other move, the point P of the latter having the velocity PG; the direction of this motion is, of course, perpendicular, not to BT, but to PD. The resultant motion of the pin connecting the sleeves must, as before, be in the direction PS; and in order to produce it there must be compounded with PG, the component of rotation, a sliding component in the direction BT:

because by the structure of the mechanism these are the only motions of which the sleeve upon the rod BT is capable. Therefore, the first partial resultant PI is found by drawing through G a parallel to BT; and the sliding along BT, by drawing through I a parallel to PG—which is perpendicular to PD.

Next, keeping BTD stationary, assign to the point P of the other rod a rotative component PL, perpendicular to PC; then by similar reasoning we find the other partial resultant PF, and sliding component PE. Compounding these two partial resultants as in Fig. 15, we obtain the final resultant PR, as the motion of the pin P when both rods move with the assigned velocities;—and the total rate of sliding along each rod is found by drawing RN perpendicular to PC, and RM perpendicular to PD.

20. We are now in a position to make the problem still more general, and to determine the motion of the point of intersection of two *curved* lines rotating about fixed centres. In Fig. 18, the rigid steel rods JK, QO, bent into the forms of circular arcs, pass through correspondingly curved sleeves pivoted together at P, and, being riveted into the arms JD, QC, are compelled to rotate about the fixed centres D, C. Through P draw a tangent to each curve; for facility of comparison, this combination has been purposely so drawn that these tangents are parallel to BT, AS, of Fig. 17, and that the points P, C, and D, have the same relative positions as in that diagram.

Proceeding as before, we assign a rotative component PG to the point P, on the arc JK, the right-hand piece being held stationary. Then it will be at once seen that the first partial resultant PI must have the direction of the tangent PS, while the sliding component must have the di-

rection PT, tangent to JK. Further explanation is needless, it being now quite apparent that the final resultant PR, and the total rate of sliding, PM and PN, are determined precisely as in Fig. 17.

21. Going one step farther, it is to be observed that the circular arcs JK, OQ, might be tangent at P to any two curues whatever, circular or otherwise; and supposing those curves to rotate about C and D, the motion of P at the instant would not be in any way affected by the substitution. It is true that if the new curves are non-circular, the device of the sliding sleeves can not be employed. But by this time it should have been perceived that they are not at all necessary any case, and that the motion of the point of intersection of the two lines, straight or curved, is in fact determined by the rotations alone.

Nevertheless, these sleeves have, as we think, been of service in the investigation, by making apparent to the eye the reasons for certain steps in the process of finding the resultant sought. And though just such combinations as these may not be met with in mechanism, yet very similar ones may be, and at any rate it has already been shown that the main question has an important bearing on some mathematical problems; nor need we go far to find instances in which its solution may be of direct practical interest.

22. For example, given the velocities of the blades of a common pair of shears, what is the rate of *shearing cut?* The same question may be asked in relation to the pruning scissors with curved blades, shown in Fig. 19. Here C is the common centre of rotation, AS is tangent to the upper blade, and BT to the lower one, at their intersection P. Regarding C as fixed, let PG, PL, be the components of rotation, both of which are perpendicular to PC. Draw

through G a parallel to BT, and through L a parallel to AS; these intersect in R, and PR is the motion of the point of intersection. This will be readily seen by the aid of the reference letters to be a case precisely similar to that shown in Fig. 18; when the blades are straight, as in the common shears, each edge is its own tangent, and the construction is made as in Fig. 17.

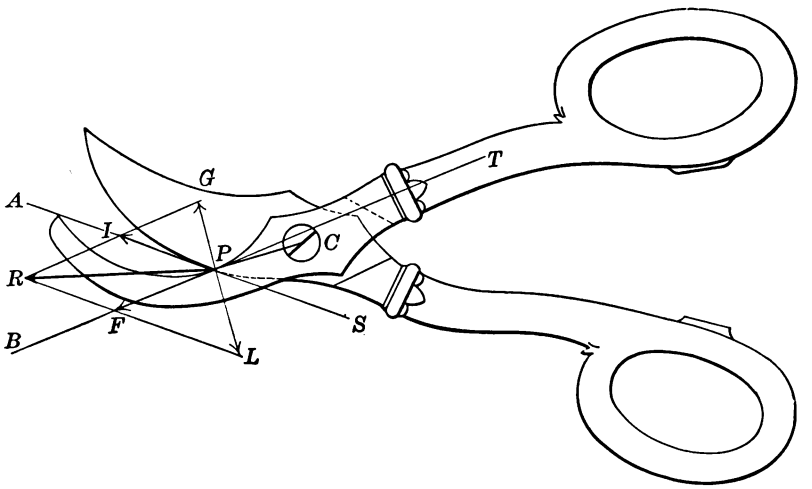


FIG. 19.

23. The direction and velocity of the motion of a point may also be determined in many cases by means of the instantaneous axis. For example: In Fig. 20, C and D are the fixed centres upon which turn the levers CA, DB; the points A and B are connected by the link AB, which is extended in a curved form to P: the motion of A is necessarily perpendicular to CA, and letting AM represent its velocity, it if required to find the velocity of the points B and P. Since B is constrained to move in a direction perpendicular to DB, the instantaneous axis is found by producing CA, DB, until they intersect in O; then drawing OM, the angle represents

the angular velocity of A around O: and this must be the same for every point of the rigid piece ABP. Also, the direction of the motion of every such point must be perpendicular to the *instantaneous radius*, or line drawn from the point in question to the instantaneous axis. Therefore, drawing through B a perpendicular to BO, and through P a perpendicular to PO, the required velocities BN, PR, are determined by making the angles δ', δ'', each equal to δ, or AOM.

In the present instance this affords, perhaps, the most

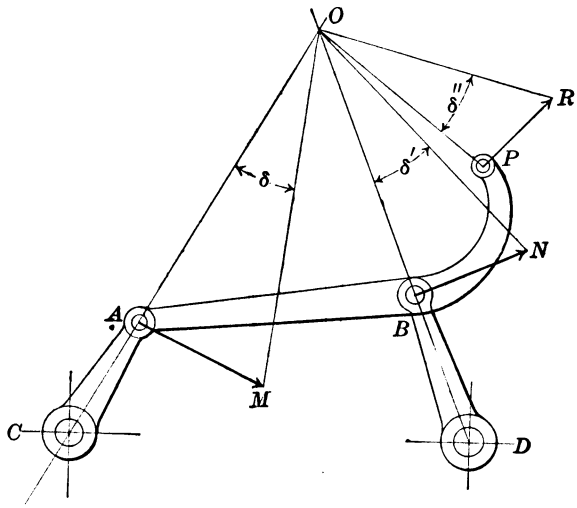


FIG. 20.

simple means of determining the motion of P; and since the direction of this motion must be tangent to the path of that point, attention is thus drawn to the fact that by the use of the instantaneous axis, a tangent to a curve may sometimes be drawn even more readily than by Roberval's method.

24. But the circumstances may be such that the instantaneous axis is inaccessible, or that its determination is un-

reliable; one or both of which things would have resulted if, in Fig. 20, the two levers had been nearly parallel. In this event other means must be employed, and in Fig. 21 a different process is illustrated, the conditions, for facility of comparison, being the same as in the preceding figure. The assigned motion AM has an absolute component Ae in the direction AB, and the motion of B must have an equal component Be' in the same direction. The actual motion of B must be perpendicular to BD, and its velocity BN is determined by drawing through e' a perpendicular to Ae'. Draw BP; then BN will have an absolute component Bg in this direction, to which Pg' must be equal. Also draw AP;

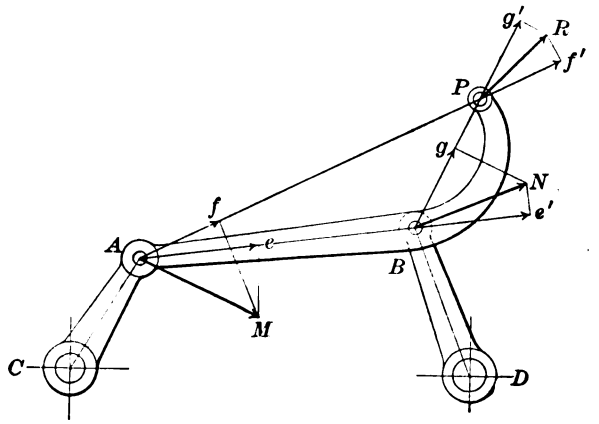


FIG. 21.

then AM has an absolute component Af in that direction, to which Pf' is equal. Through g' draw a perpendicular to Bg', and through f' a perpendicular to Af'; these intersect in R, and PR is the required motion of the point P.

25. Composition of Revolution and Rotation.— Numerous mechanical devices are practically employed, in which one wheel not only rotates about its own axis, but at the same

time travels in an orbit about another fixed axis, thus constituting what is called a planetary train. Any point connected with the planetary wheel then travels in a path dependent upon both these circular motions, and consequently called an *aggregate* path. The general principle is illustrated in Fig. 22; the wheel *W*, whose centre is *C*, has its bearings in the train-arm *AD*, which turns about *D* as a fixed centre; and this wheel may be made to turn in its bearings, independently of the motion of the arm, by any suitable means. Supposing the arm to be stationary, let the wheel rotate as shown by the arrow *x*, with such a rate that a pin *P* fixed in the side of *W* shall move with a velocity *PQ*, the direction of this motion being necessarily perpendicular to *PC*.

Next, suppose *W* not to turn at all about *C*, and let the train-arm move as shown by the arrow *y*, the velocity of *A*

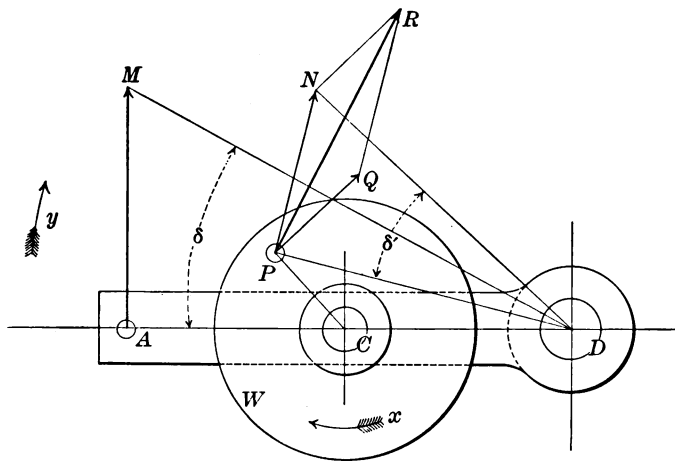


FIG. 22.

being represented by *AM*, perpendicular to *AD*. Since the arm and the wheel now move as one piece, this will impart to *P* a motion perpendicular to *PD*, the velocity *PN* being

determined by making the angle δ' equal to the angle δ. Now, if these two movements take place simultaneously, the resultant motion of P will be determined by compounding PQ and PN,—that is, by completing the parallelogram and drawing the diagonal PR; because the rotation about C and the revolution about D are wholly independent of each other, both in direction and in velocity.

26. Before proceeding to the construction of a complete velocity diagram, we propose to give a series of examples showing how the principles and processes above described may be combined and applied in determining the motions of certain points in detached mechanical movements. It is immaterial whether the combinations selected are, or are not, parts of actual machines, because it would be difficult to contrive a "mechanical movement" of practical form which might not some time be found adaptable to a useful purpose. And the study, or better still, the actual execution, of such exercises, in which attention is confined to the action of comparatively few parts, will be found the best means of acquiring thorough familiarity with the principles, and facility in their application.

27. In the first of these examples, Fig. 23, the lever CP is pivoted at C to a socket which slides along the fixed guide TT, and at A the link AQ is joined to this lever. Let AL be the absolute component of the motion of A, in the line of this link, and let CM be the resultant motion of C along the guide; it is required to find the actual motions of the points A and P. Resolve CM into the components CD in the direction PC, and CE perpendicular to it; then PH, AF, will be the absolute components of the motions of P and A in the line of the lever, each being equal to CD. Draw through L a perpendicular to AQ, and through F a perpendicular to

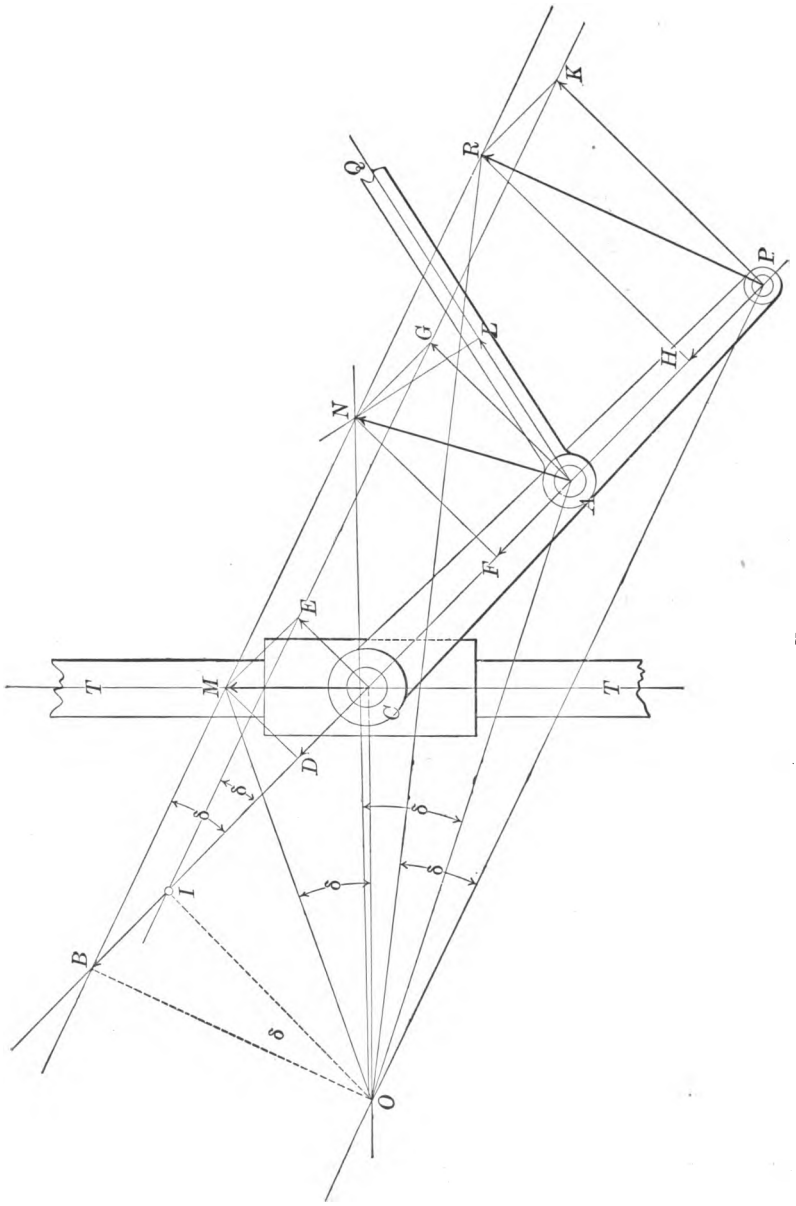


FIG. 23.

PC; these intersect in N, and AN is the required resultant motion of A. The side component of AN is AG; and the two side components AG, CE, determine a rotation of PC about the point I, where it intersects the prolongation of GE. The prolongation of EG limits PK, the side component of the motion of P, which, compounded with PH, determines PR, the required resultant.

One test of the accuracy of the construction is to draw the line RNM through the extremities of the resultants; this should be parallel to KGE, and when produced to cut the prolongation of PC in B, IB should be equal to CD.

Another test is to find the instantaneous axis of CP, by drawing, through P and C, lines perpendicular to PR and CM respectively. These intersect in O: and not only should all the angles marked δ be equal to each other, but OI should be perpendicular to PI, and OB perpendicular to RB.

28. In Fig. 24, a lever turning about the fixed centre D

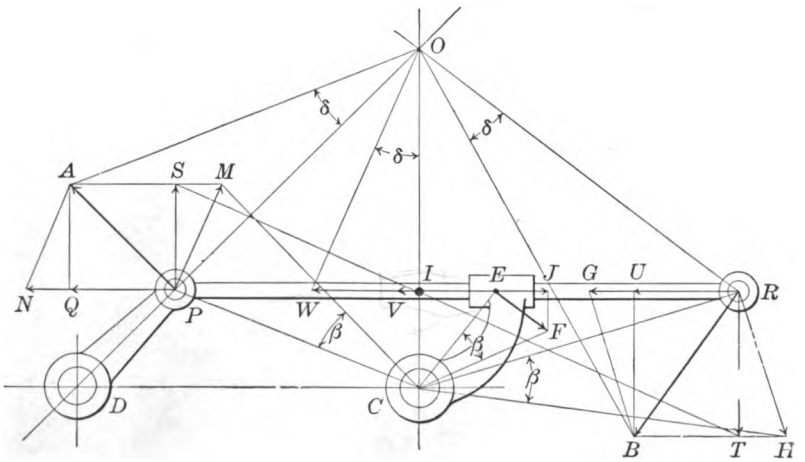


FIG. 24.

is jointed at P to a rod PR, which slides freely through a sleeve formed in the outer end of a rigid arm EC, which

turns about the fixed centre C. Given the motion PA of the point P, it is required to find the motions of R and of the point E of the arm EC, and also the rate of sliding of the rod through the sleeve.

The only motions of which the rod PR is capable, are those of revolving around the centre C, and of sliding endwise through the sleeve. We may, then, resolve PA into the sliding component PN in the line of the rod, and the rotative component PM perpendicular to PC. The former gives at once the rate of sliding, and the latter determines the angle PCM, or β, which represents the angular velocity of the rotation about C; this is the only motion of which the curved arm is capable, and EF, the required motion of the point E on that arm, is perpendicular to EC, and limited by making the angle ECF equal to β. The motion of R has also a rotative component RH, determined in like manner, and also a longitudinal component $RG, = PN$; completing the parallelogram, RB is the resultant sought.

Otherwise, the instantaneous axis of the rod PR might have been found thus: draw CI perpendicular to PR, then, in rotation about C, the motion of the point I would be perpendicular to CI, and therefore longitudinal in respect to the rod. This must be true also in the rotation about the instantaneous axis of the foot of the perpendicular from that point upon the rod; consequently, that foot can be no other than the point I, and the instantaneous axis O may thus be found by producing CI to cut the prolongation of DP. Then drawing AO and OR we may determine the absolute motions of R and of I by making the angles marked δ equal to each other.

A test of this may be applied by resolving PA into the components PQ along the rod, and PS perpendicular

to it; resolve RB in a similar manner, then PQ, IW, and RU, should be equal to each other, and ST should pass through I.

The point *E of the rod* must have an absolute component $EV = PQ$; this does not indicate the whole sliding, since EF, the motion of *E on the arm*, has a sliding component EJ in the opposite direction; the sum of these, or JV, is equal to PN, which was previously determined by another method.

29. In Fig. 25, D is the fixed centre about which turns the lever DB, jointed at B to the bar ABP. The motion of A is constrained by the lever AC, which turns about C as a fixed centre. Given the motion BM, it is required to find the motion of P.

This may be done in three different ways.

First. Resolve BM into the rectangular components BE, BF, then AF', PF'', must be equal to BF and in the same direction. The motion of A must be perpendicular to AC, is limited by $F'N$ perpendicular to AB, and has the side component AE'. A line through E' and E limits the side component PE'', and PR is found by compounding PE'' with PF''.

Second. (Same Figure.) Produce CA to cut BD in O, the instantaneous axis of AP; draw MO and PO, also Px perpendicular to PO. Construct the angle POy equal to the angle BOM; Px cuts Oy in R, and IR is the resultant sought.

Third. In Fig. 26, find BF, AN, and AE', as in the first solution. Now, the motion of P may be regarded as compounded of a rotation around B and a revolution around D. The former component is PG perpendicular to ABP, limited by the prolongation of $E'B$; the latter is PH perpendicular to PD, determined by making the angle PDH equal to the angle BDM. Completing the parallelogram, the diagonal PR is the required resultant.

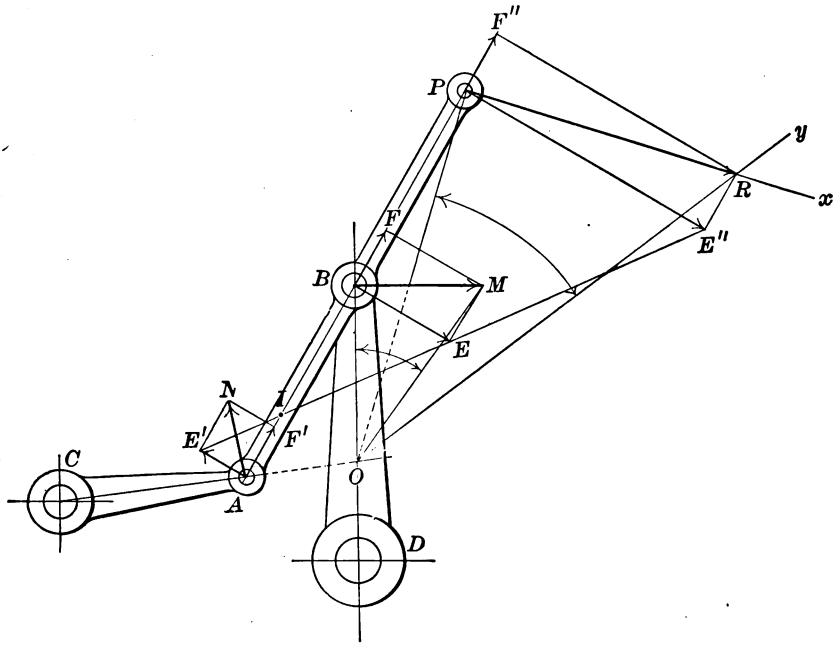


FIG. 25.

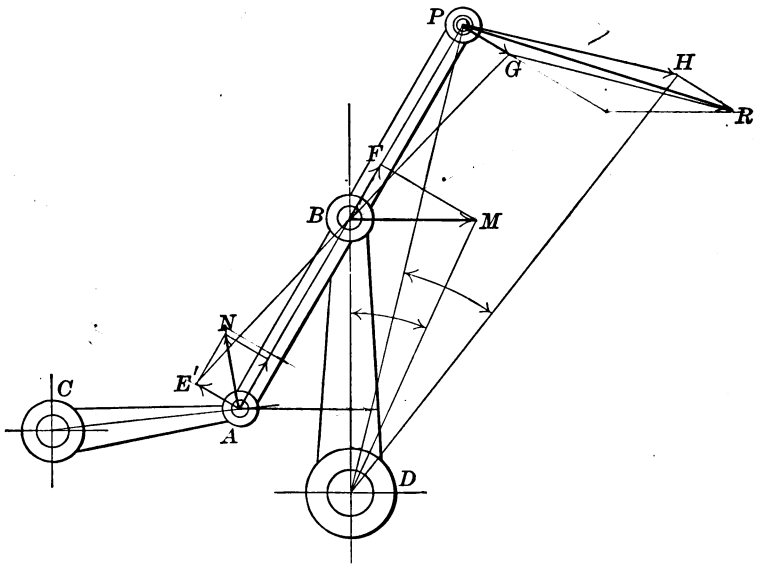


FIG. 26.

Either of these processes may be used as a check upon the other—and in practice it is often advisable to apply such tests, which may lead to the detection of an error; or, what is still more satisfactory, prove that none has been made.

30. Of Rolling Contact.—There are numerous mechanical devices in practical use, in which the motions of certain parts are determined by their connection with a curve which *rolls* upon another line which is itself stationary, and may be either curved or straight.

The nature of perfect rolling contact may perhaps be best illustrated by a study of that which is not perfect. Thus, in Fig. 27, the polygon rolls upon the fixed right line with a hobbling motion; in the position shown, the point A is at rest, and the whole figure turns about it as a centre until B comes into LM at D; it will then turn about B until the face BE coincides with LM, and so on: the perimeter of the polygon "measuring itself off" upon the straight line along which it rolls.

If the number of sides be increased, their length will be less, and the hobbling will be diminished, until, when the number becomes inconceivable, it will be come imperceptible. The broken contour then becomes the dotted curve, tangent to the base line upon which it rolls, and the change from one centre of rotation to another goes on continuously. But this does not alter the facts, that at any instant the point of contact is *at rest*, and that every point of the curve, as well as every point rigidly connected with it, is at the instant turning about that point of contact as a fixed centre.

In this case the polygon, being a regular one, ultimately becomes a circle; and when it does, the path of its centre is a right line parallel to LM. And it will now be apparent that the *rolling* of this circle upon is tangent is the resultant

of a rectilinear translation indicated by the arrow t, and a rotation about the centre C of the circle indicated by the arrow r. On LM set off AN equal to the quarter circumference ABE, and erect the perpendicular NC'; then if the circle, without rotating, be moved bodily to the right until C reaches C', the single point A of the circle will have been brought into contact with every point of AN,—and this is pure *sliding* contact. On the other hand, we may suppose C to remain fixed, and the circle to turn as shown by the arrow r through the angle ECA, or 90°; then every point of the quadrant ABE will have been brought into contact with the single point A of the tangent,—and this also is pure sliding contact.

31. But if these two motions take place simultaneously and uniformly, it is obvious that the circumference will measure itself off upon the tangent, the point 1 on the arc going to 1′ on the tangent, 2 going to 2′, and E going to N. The consecutive points of the arc, then, come into coincidence with the consecutive points of the line, each in their order of sequence; and this is pure *rolling* contact; no point of either comes in contact with more than one point of the other, and the length of the line rolled over is precisely equal to the length of the arc which rolls over it. Moreover, it is clear that the motions of the point A, due to the rotation r and the translation t, respectively, are opposite in direction and equal in velocity; they therefore neutralize each other, leaving the point of contact for the instant stationary, as previously stated.

Now, in Fig. 28, the base line LM, instead of being straight, is a part of a circle whose centre is D. Set off the arc AN equal to the quadrant AE; then if the point A of the upper circle *slides* over the arc AN, since the two circles

must be always tangent, both C and A must move with the same angular velocity about D. This being true of two points in the moving circle, must be true of all other points; the movement above supposed would therefore bring C to

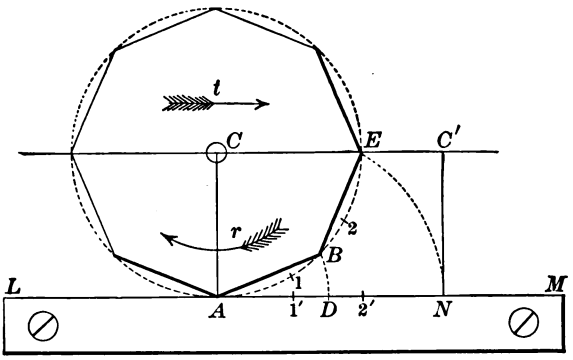


FIG. 27.

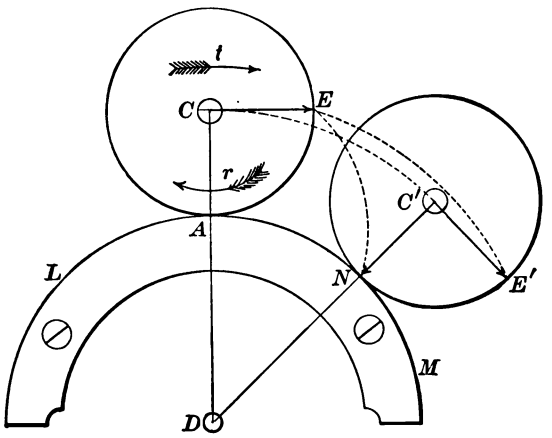


FIG. 28.

C', and E to E',—the arcs AN,CC', EE', all having the common centre D, and measuring equal angles.

32. That is to say, the motion represented by the arrow t is one of *revolution* about D, the centre of curvature of the

base line *LM*—which in Fig. 27 is one of straight translation simply because the radius of curvature of *LM* is, in that case, infinite.

In either case, the linear velocity of *A*, due to the rotation about *C*, indicated by *r*, is equal and opposite to that due to the revolution indicated by *t*; the point of contact being therefore at rest for the instant.

This fact of itself seems almost conclusive proof that the

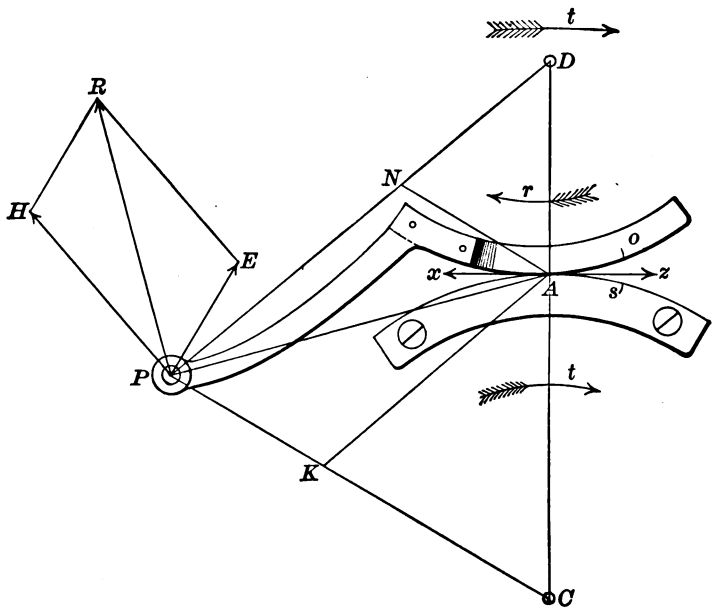


FIG. 29.

point in question is the instantaneous axis of the moving piece: this, however, is capable of still more general and rigorous demonstration, which we give here, because the fact is of importance in relation to some combinations presently to be considered, and ought therefore to be firmly established.

33. In Fig. 29, neither the fixed nor the moving curve is circular, but C is the centre of curvature of the lower one, and D that of the upper one, at their common point A; therefore CD is their common normal. Let o, s, represent points on these curves consecutive to, and equidistant from, A. Then, if the upper curve *slide* upon the lower, the point o can be brought into coincidence with A of the lower curve only by a rotation about D, indicated by the arrow r; and the point A of the upper curve can be brought into coincidence with s only by a revolution about C, indicated by the arrow t. Let the linear velocities of the point A of the upper curve, due to these motions, be represented by Ax, Az, and let these be equal to each other, since Ao, As, are equal by hypothesis.

Now let P be any point rigidly connected to the moving curve. The motion of P will be the resultant of rotations about D and C; the components are therefore perpendicular to PD and PC, respectively, and as their values we shall have

$$\left.\begin{aligned} \mathrm{PH} &= \frac{\mathrm{A}x \,.\, \mathrm{PD}}{\mathrm{AD}}, \\ \mathrm{PE} &= \frac{\mathrm{A}z \,.\, \mathrm{PC}}{\mathrm{AC}}. \end{aligned}\right\} \quad \therefore \frac{\mathrm{PH}}{\mathrm{PE}} = \frac{\mathrm{PD} \,.\, \mathrm{AC}}{\mathrm{PC} \,.\, \mathrm{AD}}. \qquad (1).$$

Completing the parallelogram, the diagonal PR is the resultant motion of P. Next draw PA, and also AN parallel to PC, and AK parallel to PD. We shall then have:

$$\left.\begin{aligned} &\frac{\mathrm{PN}}{\mathrm{PD}} = \frac{\mathrm{AC}}{\mathrm{CD}}, \ \therefore \ \mathrm{PN} = \frac{\mathrm{PD} \,.\, \mathrm{AC}}{\mathrm{CD}}, \\ &\text{and} \\ &\frac{\mathrm{PK}}{\mathrm{PC}} = \frac{\mathrm{AD}}{\mathrm{CD}}, \ \therefore \ \mathrm{PK} = \frac{\mathrm{PC} \,.\, \mathrm{AD}}{\mathrm{CD}}, \end{aligned}\right\} \quad \therefore \frac{\mathrm{PN}}{\mathrm{PK}} = \frac{\mathrm{PD} \,.\, \mathrm{AC}}{\mathrm{PC} \,.\, \mathrm{AD}}. \qquad (2).$$

The second members of Equations (1) and (2) being identical, their first members are equal; *i.e.*, $\frac{PH}{PE} = \frac{PN}{PK}$.

But the angles, *HPE*, *NPK*, are equal; therefore the parallelograms *HE*, *NK*, are similar, and the angles *RPE*, *APC*, are equal. Consequently,

EPA + *RPE*, = *RPA*, is equal to *EPA* + *APC*, = *EPC*.

Now *EPC* is a right angle by hypothesis; therefore *PR*, the absolute motion of *P*, is perpendicular to a right line drawn from *P* to the point of tangency *A*. Since *P* was chosen at pleasure, it follows that lines perpendicular to the motions of any other points connected rigidly with the moving curve, will intersect in *A*, which must therefore be the instantaneous axis of that curve: and it is perfectly clear that a rotation about *A* will bring the point *o* into coincidence with *s*, which we have already seen to be a condition of rolling contact.

34. The utility of this fact is well illustrated in Fig. 30, which shows a simple planetary train. The shaft of the wheel *W'*, which rolls around the fixed wheel *W*, has its bearings in the crank or "train-arm" *CD*, and, projecting through that arm, has secured upon it a second crank *DA*. The crank-pin *A* operates the link *AP*, of which the opposite end *P* is caused by guides, not shown, to travel in the line *CP*. The point *A* traces the epitrochoid shown in dotted line; and its position, for any given position of *CD*, is readily found as follows: starting with the two cranks coinciding in one right line *Cda*, and *m* as the point of contact, set off, on the two circles, equal arcs *mO*, *mo*; then in the rolling, *o* will go to *O*, *od* then becoming the contact radius *OD*, while the angle *oda* will remain unchanged.

The motion of *A*, for any assigned velocity of *D*, might

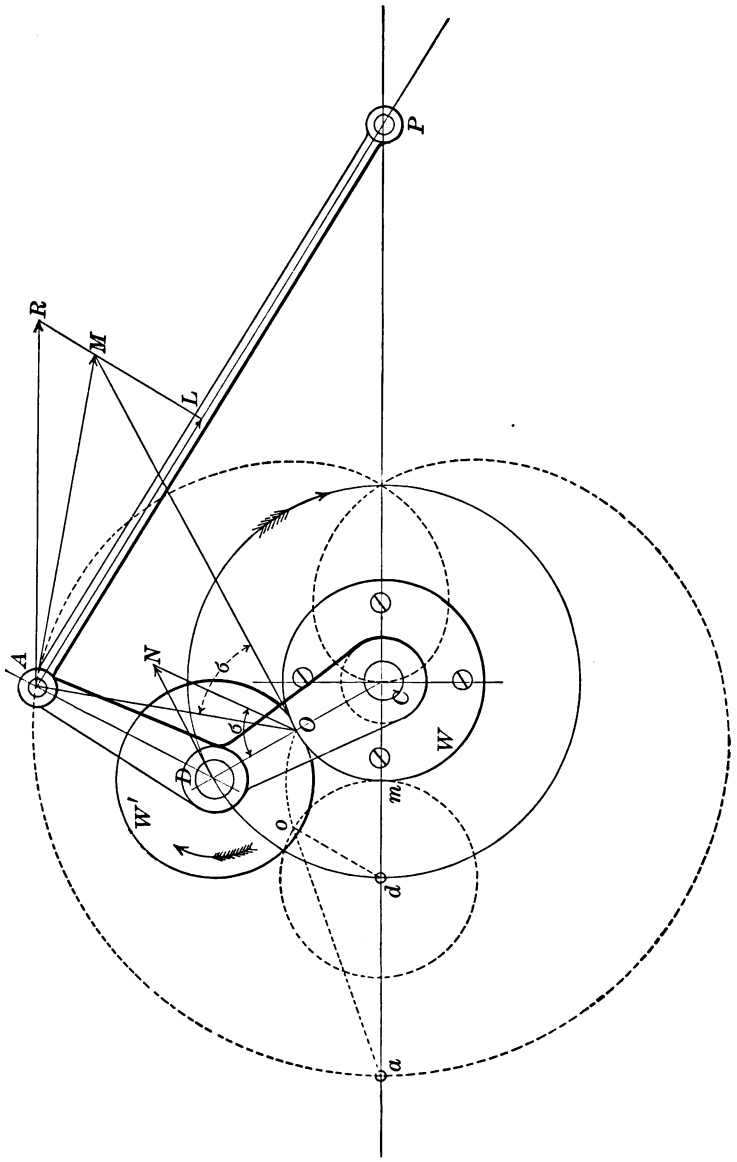


FIG. 30.

be determined as in Fig. 22, since it is the resultant of a revolution around C and a rotation about D; these are essentially independent of each other, notwithstanding the engagement of the two wheels, which is but one of many possible methods of fixing the relative directions and velocities of those two motions.

But the determination can be made in a much more simple and direct manner, because O is the instantaneous axis. Let DN be the motion assigned to D; then the motion of A must be perpendicular to the *instantaneous radius* OA, and its velocity AM is found by making the angle AOM equal to the angle DON.

If it be further required to find the velocity of P, the argument is, that on the prolongation of AP a component should be set off equal to AL the absolute component of AM, and at its extremity a perpendicular to AP should be erected, which would cut CP produced, at a point limiting the required resultant. This, obviously, would result in the construction of a right-angled triangle similar and equal to ALR, formed by drawing through A a parallel to CP, cutting ML (produced if necessary) in R; which gives AR as the required velocity of P. This abbreviation may evidently be employed in any case where the motion of one end of a link is given, and that of the opposite end in a given direction is required.

35. Fig. 31 also exhibits a simple planetary combination, in which, however, the central (or sun) wheel W is not fixed, but turns freely about the axis C. The planet-wheel W' is rigidly secured to the connecting-rod LP, the point P being made to travel in the line CP; and the two wheels are kept in gear by means of the link CD. This arrangement is known as "Watt's sun-and-planet wheels," having been em-

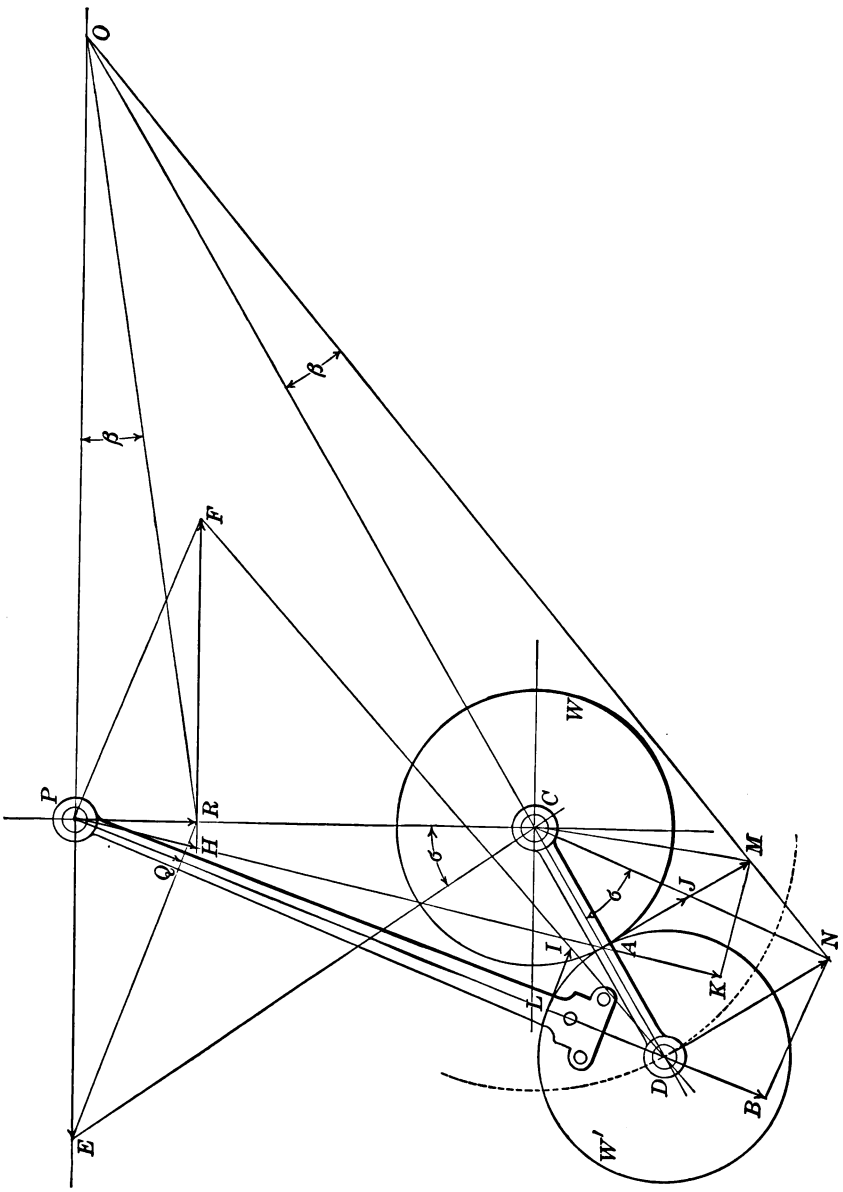


FIG. 31.

ployed by the illustrious engineer as a substitute for the crank, upon which, as connected with a rotative steam engine, some one had secured a patent.

If, now, we assign to D a velocity DN, the motion of P is ascertained, as usual, by making PQ equal to DB, the absolute longitudinal component of DN, and drawing QR perpendicular to DP. It is next to be noted that the motion of W' consists of a revolution about C and a rotation about its own centre D; and the same is true of the motion of P, since it is rigidly connected with that wheel. Resolving PR accordingly, PE is the component of revolution, and PF that of rotation. The angular velocity of the revolution is represented by the angle PCE, which of course is equal to DCN; and AJ is the linear velocity of A, due to this motion. The angular velocity of the rotation is represented by the angle PDF, and LI is the additional circumferential velocity due to that motion, to which JM is made equal, giving AM as the actual velocity of A: and ACM represents the angular velocity of W.

36. A test of the accuracy of both the analysis and the construction may be applied by producing DC to cut the prolongation of EP in O, the instantaneous axis of the planet-wheel and its attached rod. Draw NO and RO; then the two angles marked β should be equal, and NO should pass through the point M: also it is to be observed that when O is accessible, this affords the readiest means of determining AM when DN is given, and *vice versa*.

It is quite apparent that if the velocity of D is constant, the velocity of W will vary, unless DP is infinite, W' then having a motion of circular translation; on the other hand, if W turns uniformly, the motion of D will be variable. Consequently the law of variation in the piston speed will

be different in an engine in which this arrangement is used, from that obtaining in one provided with an ordinary crank, the main shaft revolving uniformly in each case. If the circumferential velocity of W be assigned, as AM, the velocity of P may be found thus: draw PA and produce it, find the absolute component AK along that line, make PH equal to it, and draw HR perpendicular to PA; otherwise find the instantaneous axis O, and make the angle POR equal to the angle AOM.

37. In Fig. 32, the engaging wheels of which W, W', are the pitch circles, turn about the fixed centres D and C. In the front face of W' is fixed a pin B, turning freely in a block, which slides in a slot formed in one arm of a bent lever; this lever turns about the fixed centre E, and in the extremity F of the other arm is a pin upon which is hung a swinging lever FP: from the front face of W projects a pin A, connected with the free end of the swinging lever by a link AP. Considering A as the driving point, and assigning to it a motion AM (necessarily perpendicular to the radius DA), it is required to determine the motion of P.

Produce DA to J on the circumference of W, at which point draw a tangent, limited by its intersection at L with the prolongation of DM; JL thus determined is the circumferential velocity of W. That of W' is necessarily the same; therefore, producing CB to cut the circumference at J', draw the tangent $J'L'$ equal to JL; draw CL', and at B draw a perpendicular to CB, cutting CL' in N; then BN is the linear motion of B. Resolve BN into two components, one in the line BE, the other perpendicular to it; the longitudinal component merely produces a sliding of the block in the slot, but the side component Bd establishes a rotation of the bent lever about E, of which the angular velocity is

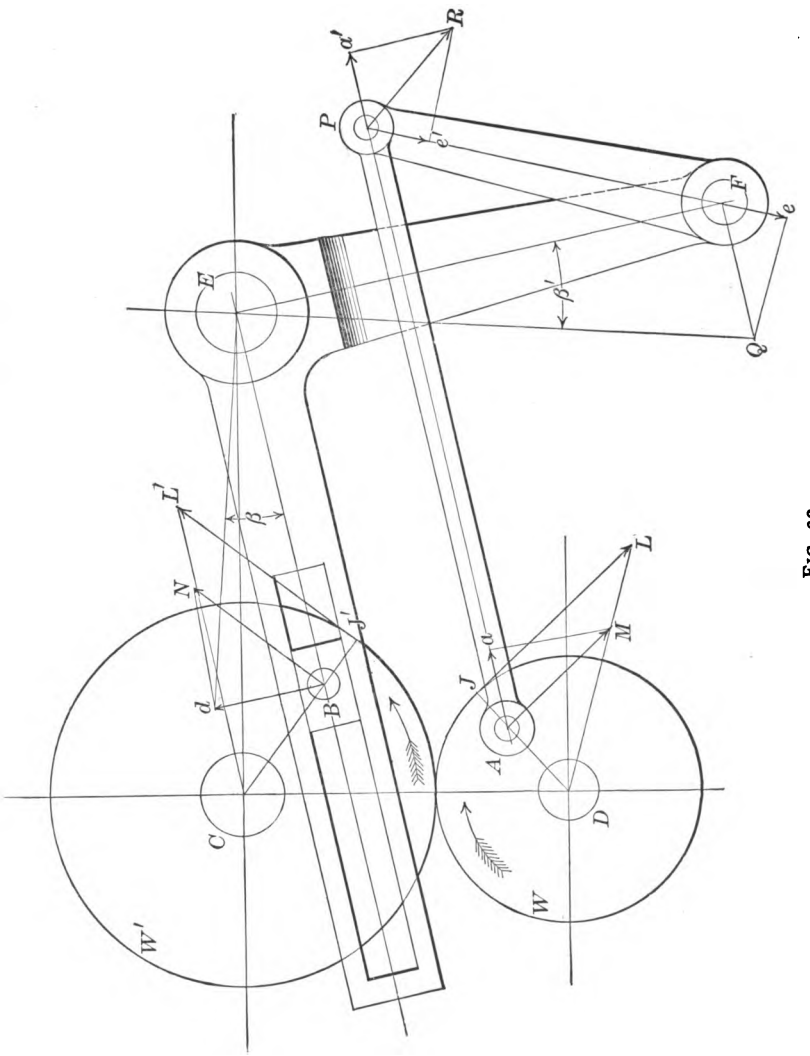


FIG. 32.

represented by the angle β, or BEd. The point F must have the same angular velocity, and must move in a direction perpendicular to EF, so that its linear velocity FQ is determined by constructing the angle β' equal to β, and the absolute component of FQ along PF is Fe, to which Pe' must be equal.

Returning now to the pin A; its motion AM has an absolute component Aa in the line AP, to which Pa' must be equal. Then, erecting at a' a perpendicular to AP, and at e' a perpendicular to FP, the intersection R of these perpendiculars is the extremity of the required resultant motion PR.

38. In Fig. 33, the planet-wheel W', which rolls inside the fixed annular sun-wheel W, has its bearings at the extremity C of one arm of a bent lever CDE, which turns about D the centre of W. At the extremity E of the other arm of this lever, is the bearing of a pin which projects from the farther side of the arm or lever EF, and is made in one piece with it. The lower end of this arm is formed into a sleeve, through which slides freely the rod BS, perpendicular to EF; this rod is formed, at the left-hand end, into an eye, fitted upon a pin B, which is fixed in the planet-wheel and projects from its front face. Finally, to the free end of EF is pivoted the link or connecting-rod FP, of which the farther end P is made to travel in the line PD.

Suppose for the moment the annular wheel W to be removed, and the bent lever CDE to be stationary, while W' revolves about C as a fixed centre. Then it is to be noted that the kinematic action of the virtual crank-arm CB and the bent lever BEF is precisely the same in this figure as in the preceding one; in this case the sliding in the direction BE occurs in the sleeve at E, whereas in Fig. 32 it

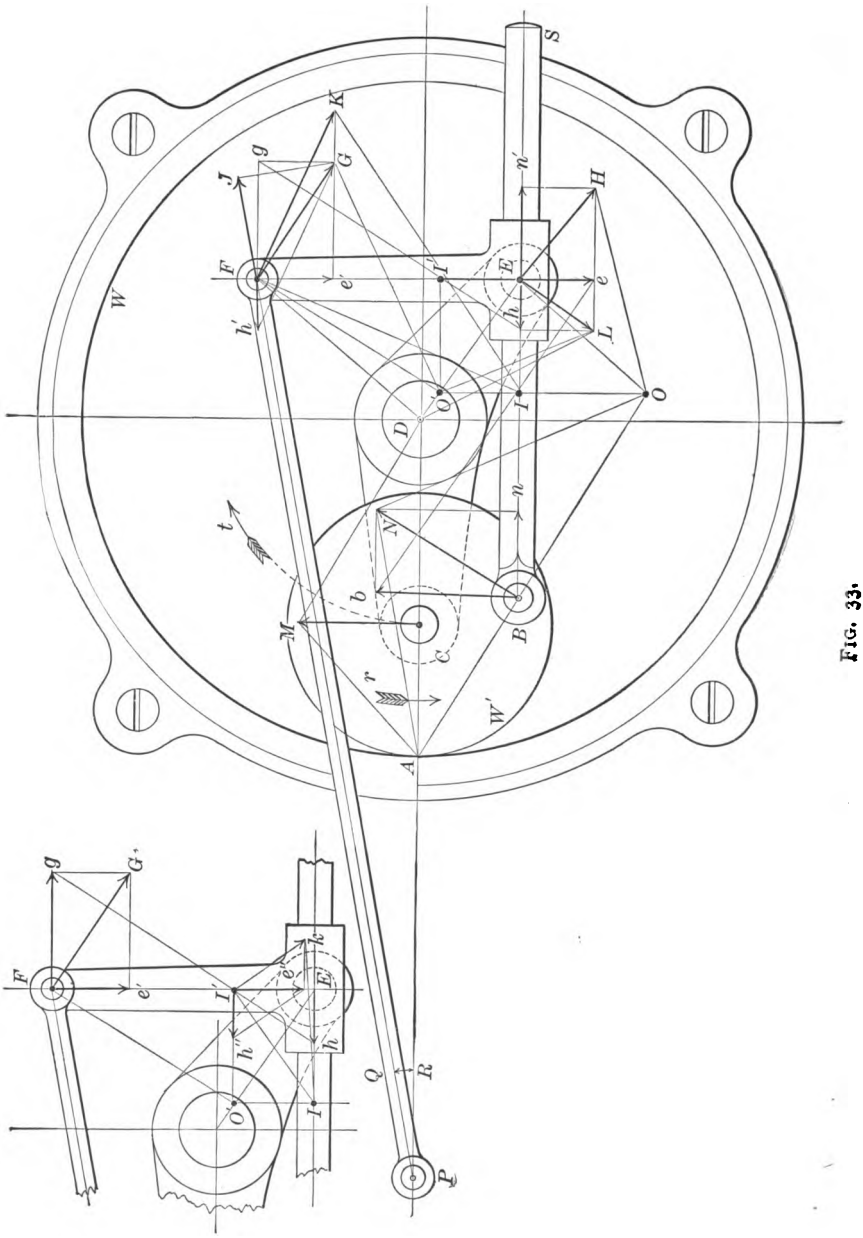


FIG. 33.

was accommodated by the motion of the block at B in the slotted arm BE. In either case, then, the rotation of B about C causes F to vibrate in a limited arc about E. Now when W is replaced and CDE is made to revolve about D, this vibration goes on as before, so that the length of the virtual crank-arm DF, by which P is actuated, is continually varying within certain limits. Let C turn about D as indicated by the arrow t, then W' will rotate about C in the direction shown by the arrow r.

39. Assign to C a velocity CM, then its angular velocity about the instantaneous axis A will be represented by the angle CAM; the motion of B will be perpendicular to the instantaneous radius AB, and its linear velocity BN is determined by making the angle BAN equal to CAM. The motion of E is EL perpendicular to DE, the angle EDL being made equal to the angle CDM. The absolute components of BN are Bb, Bn; those of EL are Ee, Eh. Join the extremities of the side components by the line be, cutting BE in I. This point is the instantaneous centre about which BS is rotating; and by reason of the connection between EF and BS, it is clear that EF must also rotate about I, in the same direction and with the same angular velocity. Therefore the motion of F has a component of rotation FK perpendicular to FI, the angle FIK being made equal to the angle EIe; and also a component of translation Fh' parallel and equal to Eh. Completing the parallelogram, the motion of F is the diagonal FG. This has a component FJ in the line of PF, to which PQ is equal; then draw at Q a perpendicular to PF cutting PD, the line of travel, in R, and PR is the resultant motion of P.

40. Now, the point E is common to BS and EF, and both pieces have the same instantaneous centre I; consequently

their instantaneous axes must lie in a perpendicular to BS through I. This perpendicular cuts the prolongation of AB in O, which is the instantaneous axis of BS; and it cuts DE in O', the instantaneous axis of EF. The actual motion of the point E on BS is EH, the resultant of the components Ee, and En' equal to Bn, and OE is perpendicular to EH; (incidentally, the sliding at E is equal to $Eh + En'$): also, $O'F$ is perpendicular to FG, the motion of F.

Again, the actual motion in space of the line EF, is determined by the two motions EL, FG; the absolute components of EL, as above seen, are Eh, Ee,—those of FG are Fg, Fe'. Draw gh cutting EF in I'; then it is seen that the motion of EF may be regarded as a rotation about I' as an instantaneous centre, combined with a longitudinal translation in the line EF.

41. And this motion is equivalent to a rotation about an instantaneous axis, which must lie in a perpendicular to EF through the point I'; this perpendicular must, therefore, pass through O', which has already been shown to be the instantaneous axis. And since all the longitudinal components must be equal, it follows that $Fe' = Ee$. Also since at I' there is no side component, the absolute motion of that point is in the direction FE, and equal to Ee. In the small diagram at the left, this motion, $I'e''$, is seen to be the resultant of a rotative component $I'k$, perpendicular to II' (the angle $I'Ik$ being equal to the angle EIe), and a component of translation $I'h''$, equal and parallel to Eh.

42. A peculiar device for producing aggregate motion, the action of which is at first glance rather obscure, is shown in Fig. 34. It consists of a crank AD, turning about the fixed centre D; a connecting-rod AB, and a lever BC, which vibrates about C as a fixed centre. Upon the pin B are

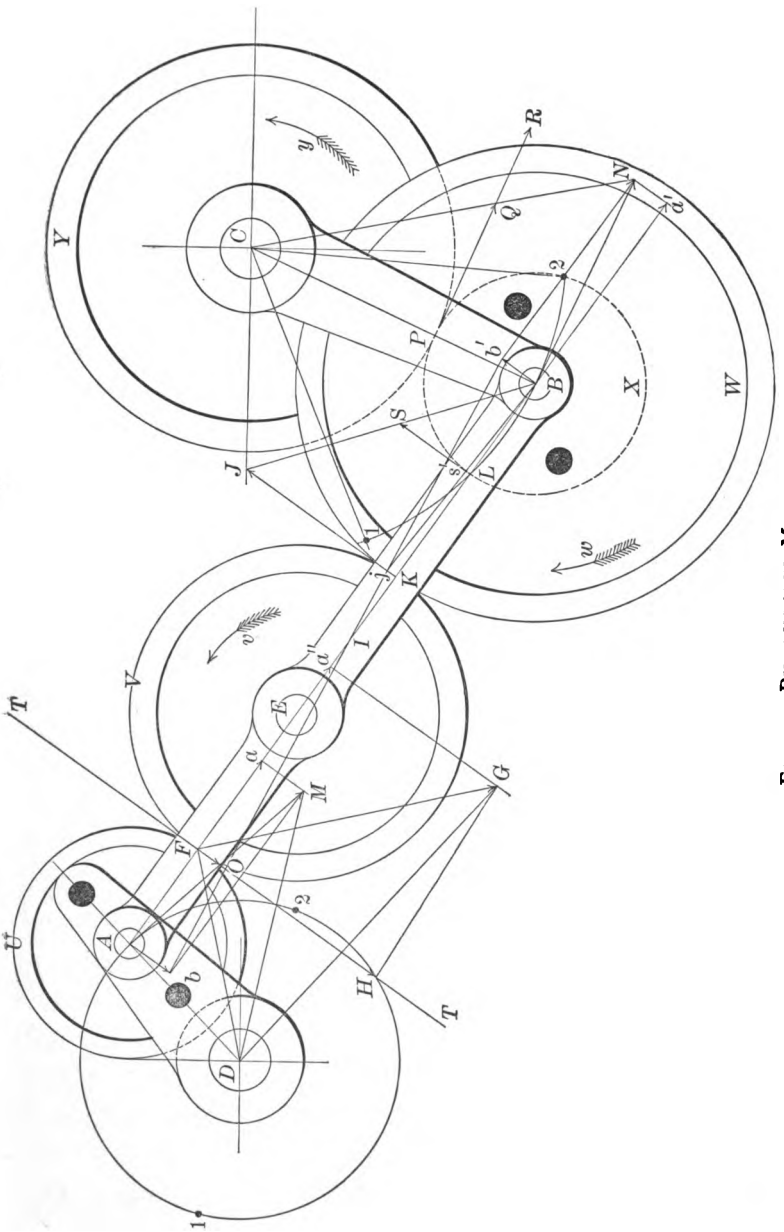


FIG. 34. PILGRIM-STEP MOTION.

hung the wheels W and X; these are secured together, and turn freely on the pin, and X engages with a wheel Y, which turns freely about its centre C.

A wheel U is secured to the crank AD, and communicates motion to W through the intervention of an idle wheel V, whose bearing is the pin E, fixed in the connecting-rod.

In this arrangement the crank AD is the driver, and the ultimate follower is the wheel Y; given the velocity of the crank-pin A, then, the problem is to find the circumferential velocity of that wheel.

It needs no argument to show that if this problem be attacked analytically, any formula expressing the action of such a train will be so complicated as to render the solution tedious, if not difficult; but if treated graphically it is both simple and easy, and requires only the application of principles already explained.

Supposing the crank to turn to the right, let AM represent the motion of A. Resolve AM into the rectangular components Aa, Ab, and set off Ba' equal to Aa; draw at a' a perpendicular to AB, and at B a perpendicular to BC; these intersect in N, and BN represents the motion of B. At P, the point of contact between the circumferences of X and Y, draw PQ perpendicular to BC, and limited by its intersection with CN. Suppose for the moment the wheel W to be removed; there would then be nothing to cause X to rotate about B, so that X, Y, and BC would move as one piece, turning about C, and PQ would represent the circumferential velocity of the wheel Y.

But when W is restored, this value PQ will be affected by two things; of which the more important is due to the rotation of V about its centre E, caused by its engagement with U, which is secured to the crank AD.

In order to determine the velocity of this rotation, draw TT, tangent to U and V at their common point F, and join bb' the extremities of the side components of AM and BN. Then bb' cuts TT in O, and FO is the tangential component of the point F on the circumference of V, supposing U to be removed. But F is also a point on the circumference of U, and its absolute motion when so considered is FG, perpendicular to FD, and limited by constructing the angle FDG, equal to the angle ADM. Draw GH perpendicular to TT; then FH is the tangential component, and OH is the circumferential velocity of V, in its rotation about E—the direction being as indicated by the arrow v. This rotation will impart to W an equal circumferential velocity; therefore we make KJ, the common tangent of V and W, equal to OH. Draw JB to limit LS tangent to X and parallel to JK, then LS should be in this case added to PQ on account of the rotation of V about its centre E.

43. But this is not all; as previously stated, the value PQ is affected by another thing, which is due to the motion of the link AB (carrying V with it), in relation to BC. This will be clear when it is considered that if V did not turn at all about E, but moved as though fixed to the link, the point K on its circumference would still have a motion Kj, determined by the intersection of bb' with KJ, which also would be imparted to W. Draw jB cutting LS in s; it will then be clear that PQ must is this case be increased by an amount QR equal to $LS + Ls$, which gives PR as the circumferential velocity of the last wheel in the train.

It is apparent that the motion Ls is due to the folding up of the link AB upon the lever PC; which will continue until the lever reaches the position $C2$, the extreme limit of its outward excursion. On the return, the directions of

both PQ and Ls will be reversed; and the actual velocity of P will be $PQ + Ls - LS$, since the rotation of V about E goes on continuously in the same direction.

It is also evident that LS is always greater than Ls; so that, while the wheel Y turns first in one direction and then in the other, it is driven farther to the left than to the right during each reciprocation of the lever BC, and consequently it will ultimately make complete revolutions about its centre C, in the direction of the arrow y.

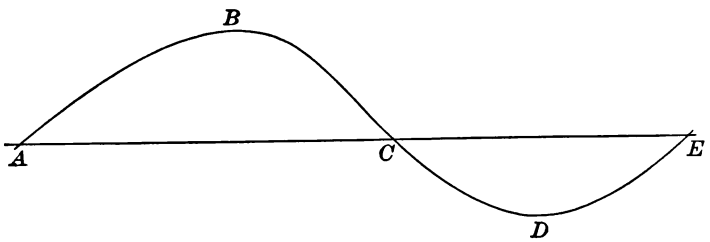
Now, because the motion of a point in the circumference of Y is reciprocating, and of less velocity in one direction than in the other, it follows that a "velocity diagram" representing the motion will be of the general form $ABCDE$ in Fig. 35; the point A representing the instant when the wheel begins to turn in the direction of the arrow, the point C the instant of reversal, and the point E the instant when it again comes to rest. Since all this is accomplished during one revolution of the driving crank, AE represents the time occupied by that revolution, of which the velocity is uniform.

44. Obviously it is desirable that such a diagram should begin and end at zero. But in this case we are confronted at the outset by the question, at what position of the crank *is* the wheel at rest? Clearly it is easy enough to determine the dead centres 1, 2, of the crank, and the corresponding positions 1, 2, of the pin B; at which instants the lever BC is at rest: but at either of those instants the wheel Y will be found to have a definite velocity, which may be determined by the processes above set forth.

In such a case the construction of the velocity diagram is the readiest means of determining the required dead points of the driven wheel. In Fig. 36, make FL equal to

AE of Fig. 35; place the crank in the position 1D, the lever in the position 1C, determine the velocity of P, and set up the ordinate FG equal to it; then the ordinate LM will plainly be equal to FG. Without entering into the details of the construction, suppose the velocity diagram $GHJM$ to

FIG. 35.



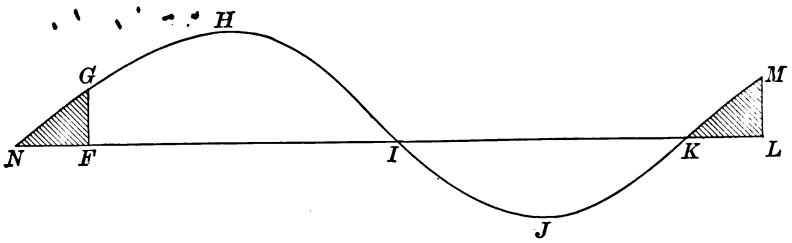


FIG. 36.

be drawn; this curve will cut FL at the points I, K, which represent the instants when Y is at rest.

Then supposing the crank-pin A, in Fig. 34, to start from point 1 upon its circular path (which corresponds to F in Fig. 36), divide the circumference of that path into parts proportional to FI, IK, KL. The points of division corresponding to I and K will then be the positions of the crank-pin at the instants when the wheel Y is at rest; and the positions of the lever BC at those instants can then be readily found in the usual manner.

Now if, in Fig. 36, we set back FN equal to LK and copy

the portion *MK* in the position *GN*, we shall have the curve *NHIJK*, the required diagram, identical in form and arrangement with *ABCDE* in the preceding figure.

45. A further example of planetary wheel-work forms part of the combination shown in Fig. 37.

The central, or "sun," wheel *W* is stationary, as shown by the screws securing it to the frame.

The shaft *D* of this wheel turns freely in its bearings, and an eccentric is keyed upon it, in which are the bearings of the shaft of the planet-wheel *W'*, whose centre is *C*; thus the eccentric itself forms the train-arm. The shaft *C* has keyed upon it the wheel *W'* at the back, and the crank *CE* is in front of the eccentric; the eccentric-rod is pivoted at *B* to a sliding socket which moves upon guides; and this socket carries a pin *F*, upon which turns one end of the lever *FP*; and this lever is connected with the crank *CE* by a link *EH*. Now, supposing the eccentric to turn about *D* at a given rate, it is required to determine the direction and velocity of *P*'s motion at the instant when the parts occupy the position shown.

The centre of the eccentric is *A*; let *AM* be the motion assigned to it; then *CM'*, the velocity of *C*, is found as shown by prolonging *DM*.

Now, the crank *CE* and the wheel *W'* being virtually one piece, are at the instant rotating about the instantaneous axis *I*, the point of contact between the sun-wheel and its planet.

Therefore, drawing *IM'*, *IE*, and *EL* perpendicular to *IE*, then *EL*, limited by making the angle *EIL* equal to the angle *CIM'*, will be the motion of *E*; this has a component *Eg* in the direction of *EH*, to which *Hg'* in the same direction must be equal. Draw *AB* and produce it, making *Ba'*

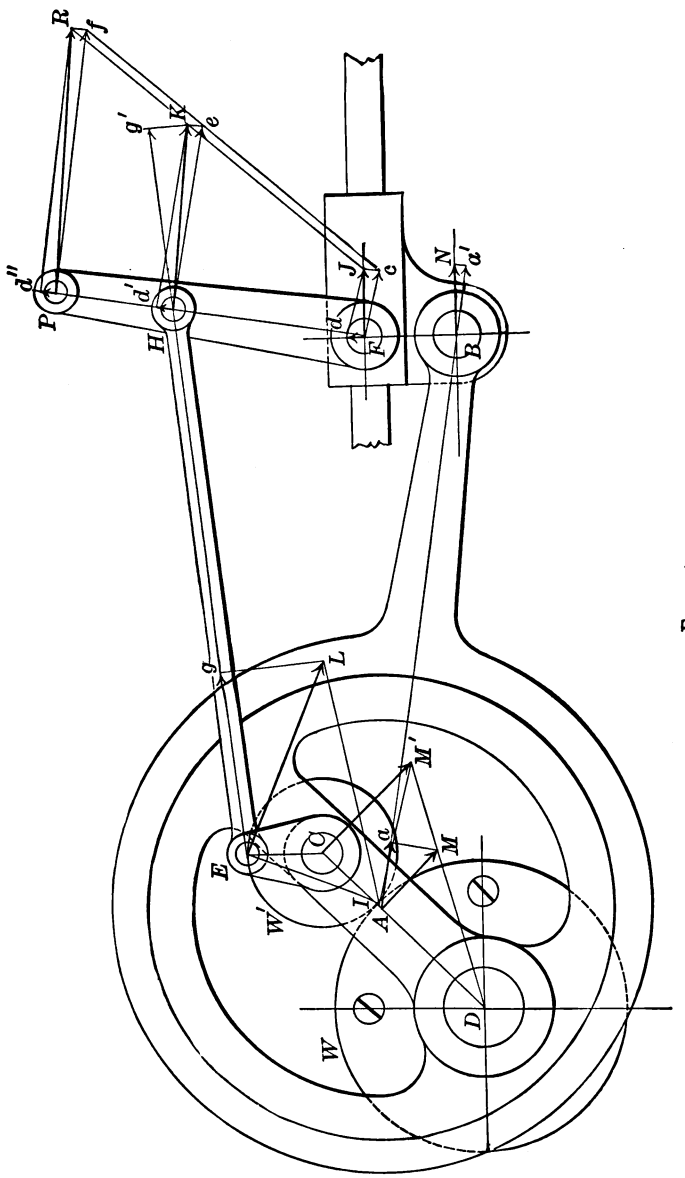


FIG. 37.

equal to *Aa*, the component of *AM* in the line *AB*; and draw *a'N* perpendicular to *Ba'* to determine *BN*, the motion of *B*, to which the motion *FJ* of the point *F* is parallel and equal. Then, with reference to *FP*, *FJ* has the side component *Fc*, and the longitudinal component *Fd*; and both *Hd'* and *Pd''* in the direction *FP* must be equal to *Fd*.

Draw at *d'* a perpendicnlar to *FP*, and at *g'* a perpendicular to *EH*; these intersect in *K*, and *HK* is the resultant motion of *H*, which has a side component *He*. The prolongation of *ce* will limit the side component *Pf* of the motion of *P*; the longitudinal component is *Pd''*, and the diagonal *PR* of the completed rectangle is the required resultant motion; and *JKR* will be a right line parallel to *cef*, if the construction has been correctly made.

46. In Fig. 38, *W* is the pitch circle of a wheel formed in one piece with an eccentric keyed upon the shaft *D*; for the sake of avoiding confusion, this pitch circle is drawn of the same diameter as the eccentric.

As in the preceding figure, the eccentric rod is pivoted at *B* to a piece which slides upon guides, and carries a projecting pin *G*, upon which turns the lever *GP*. Then *A* being the centre of the eccentric, and *AM* representing its motion, the motions *BN* and its equal *GV* are determined as in Fig. 35, so that no more need be said of them.

The eccentric rod in this case has a projecting portion on the upper side, in which is fixed a pin *C*, with a wheel *w* turning freely on it; and *w* engages with *W*. The smaller wheel has also a projection from its front face, in which is fixed a pin *E* by which a link is pivoted to it, whose other end is pivoted at *P* to the lever *GP*. Then, knowing *AM*, we are to ascertain the motion of *P* in velocity and direction. To begin with, since *GV* is already known, we find

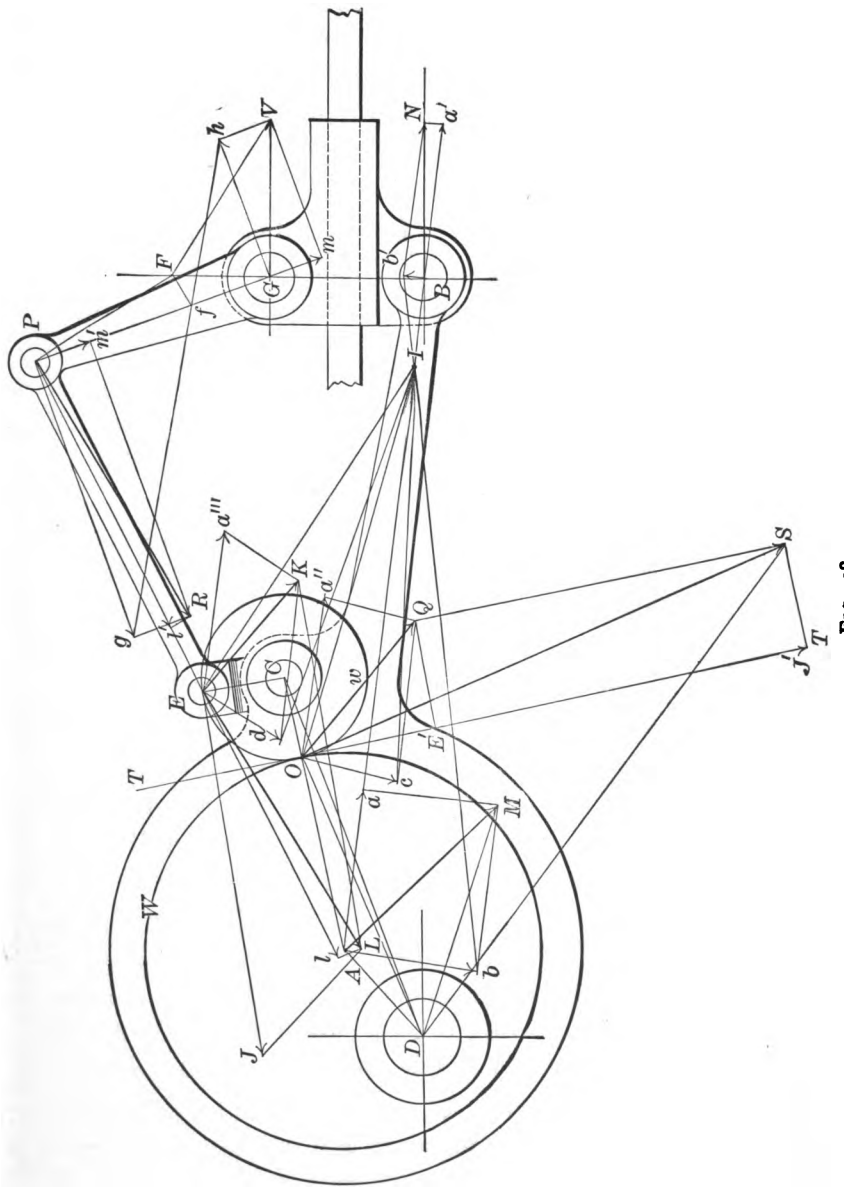


FIG. 38.

with reference to GP its side and longitudinal components, Gh and Gm; then, since Pm' must be equal to Gm, it remains to find the absolute component of the motion of P, or, what is the same thing, of E, in the line of the link EP.

Now, considering the wheel w by itself, it turns on its bearing at C solely by reason of its engagement with W; so that, if the latter were removed, it would move as one piece with the eccentric rod. And the motion of that rod consists in a rotation about I (the intersection of AB with the line bb' joining the extremities of the side components of the motions of A and B), combined with a translation in the direction AB,—and the magnitude Aa of the latter is already known.

The wheels W and w touch each other at the point O on AC; then Oc, perpendicular OI, of such length that the angle OIc is equal to the angle AIb, is the rotative component of the point O on the connecting-rod; the component of translation is Oa'' equal and parallel to Aa, and OQ, the diagonal of the completed parallelogram, is the resultant motion, which would be the same for the point O on the smaller wheel were the larger one removed.

47. But the motion of the point O on the wheel W, is OS perpendicular to OD, limited by making the angle ODS equal to the angle ADM. Now draw TT, the common tangent to the two wheels, and upon it let fall the perpendiculars QE', SJ'; these, of course, should be equal to each other and to the normal components of the motions OQ, OS. The tangential components OE', OJ', lie in the same direction, so that their difference EJ' represents the circumferential velocity of the rotation of w about C. For the sake of perspicuity we have placed the point E upon the circumference of w, so that EJ, perpendicular to CE, and equal to $E'J'$, is

one component of the motion of E, and is due solely to the engagement of the two wheels. But were W removed, E would be to all intents and purposes a point of the connecting-rod; and its motion might be determined (since the instantaneous axis is inaccessible) just as OQ was, which is the method here exhibited; Ed perpendicular to EI, making the angle EId equal to the angle AIb, is one component, while Ea''' equal and parallel to Aa, is the other; then EK the diagonal of the completed parallelogram is the resultant motion upon this supposition. Finally, complete the parallelogram $JEKL$, and its diagonal EL is the resultant motion of E when the whole mechanism is set in action; this has a component El in the line PE, to which Pl' must be equal; erecting at l' and m' perpendiculars to PE and PG, respectively, they intersect in R, and PR is the resultant required.

In reference to PG, the side components at P and G are Pg, Gh; draw gh cutting PG in f, and at this point erect a perpendicular to PG; this should pass through F, the instantaneous axis of PG, which is the intersection of PF, GF, respectively perpendicular to PR and GV.

SIMULTANEOUS DEAD-POINTS IN LINK-WORK.

48. In the train of link-work shown in Fig. 39, the driving crank DA, by means of the connecting-rod AB, imparts a vibratory movement to the lever CE; which, in its turn, causes the bent lever FGH to vibrate through the intervention of the link EF; finally, to this bent lever is pivoted the link HP, whose extremity P is constrained by guides (not shown) to travel in the vertical line IL. And the problem is, assigning any velocity to the crank-pin A, to determine the resultant velocity of P.

Now this combination, consisting as it does of simple levers and links, presents no new feature, except in the two critical positions when the crank-pin A reaches either a

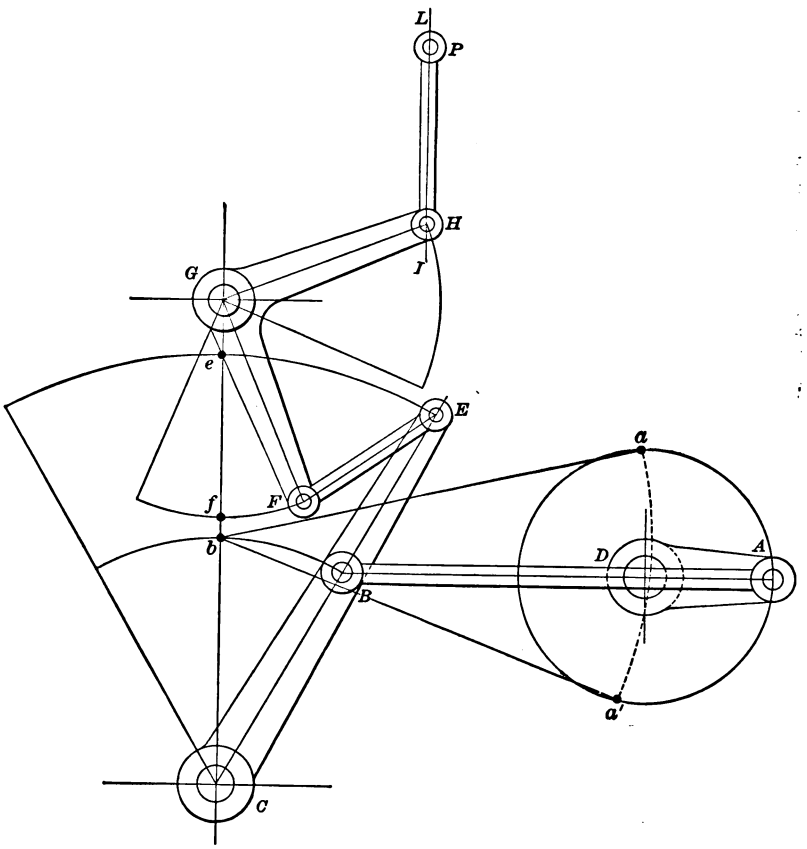


FIG. 39.

or a'. The proportions here given are such that $CG = CE - EF + FG$; so that when A reaches either of the above-mentioned points, the centre lines CE, EF, and FG, coincide in one line $CfeG$: the link being thus folded up upon both levers, we have two simultaneous dead-points.

It is stated by Prof. Rankine ("Machinery and Millwork," p. 193), that in these circumstances the ratio of the angular velocities of the two levers is *indeterminate;* were this so, then the ratio between the linear velocities of the points E and F would also be indeterminate. I propose to show that neither of these things is true.

Since the motions of E and F, in the position under consideration, are both perpendicular to the line joining those points, the motion of the link must, at the instant, be one of rotation about some point on EF or its prolongation, and the first step is to find that point; the method of doing this

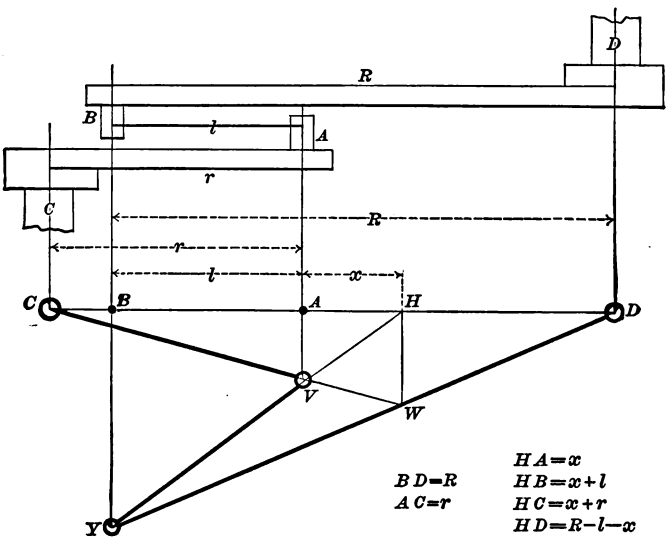


FIG. 40.

is illustrated in Fig. 40. The upper part of this figure gives a side view of two levers CA, DB, connected by a link AB, at the instant of collapse. In the movement diagram below, it must be understood that CV, DY, do not represent *positions* of the levers, but that AV, BY, and HW, represent

the *motions* of the points A and B, and of H the required instantaneous axis, at the critical instant.

The actual magnitudes of these lines are immaterial, but the relative velocities must be such, and H must be so situated, in all cases, as to satisfy the following conditions, viz.:

1. Because the link AB is, at the instant, turning about H as a centre, the line VV, or its prolongation, must pass through H.

2. Because H is the virtual intersection of the centre lines of the levers AC and BD, the magnitude of HW must be such that the prolongations of CV and DY shall pass through W.

49. The lines of this diagram form three pairs of similar right-angled triangles, from which the following proportions are readily deduced, viz.:

$$\left.\begin{aligned}\frac{AV}{HW}&=\frac{AC}{HC}\\ \frac{HW}{BY}&=\frac{HD}{BD}\end{aligned}\right\}\quad\therefore\ \frac{AV}{BY},=\frac{HA}{HB},=\frac{HD}{HC}\times\frac{AC}{BD}.$$

Or, substituting the symbols given in the figure,

$$\frac{x}{x+l}=\frac{R-l-x}{x+r}\times\frac{r}{R};$$

whence

$$x^2-\frac{2rl}{R+r}x=\frac{Rlr-rl^2}{R+r},$$

and this gives

$$x=\frac{\pm\sqrt{Rlr(R-l+r)}-rl}{R+r}.$$

Which proves that the distance of the instantaneous axis H from the point A is not indeterminate, but must have one

of two definite and easily ascertained values: and the ratio of the linear velocities of the points A and B is the same as the ratio of their distances from the instantaneous axis,—*i.e.*,

$$\frac{AV}{BV} = \frac{HA}{HB}.$$

In the figure the proportions are as follows: $R = 6$, $r = 3$, $l = 2.25$, and the resulting values are, $AH = +\ 1.087$, $AH = -\ 2.587$. The first value is the one taken in Fig. 40; the result of taking the second is shown in Fig. 41. It is

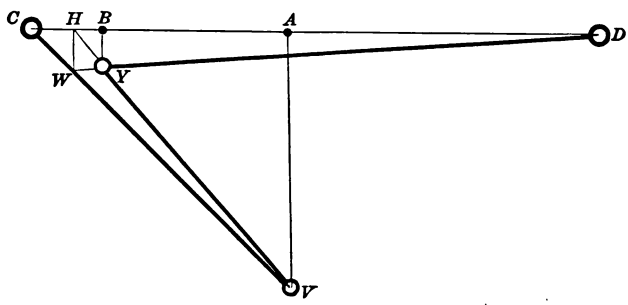


FIG. 41.

apparent that the diagrams can not be constructed with certainty until these values have been computed; though the rudest sketch will serve the purpose in making the calculations.

50. The ratio of the linear velocities being, then, determinate, and the lengths of the levers fixed, the ratio of their angular velocities must also be determinate. In order to find its value, let v represent the angular velocity of R, v' represent the angular velocity of r; then, since angular velocity $= \frac{\text{linear velocity}}{\text{radius}}$, we shall have

$$\left.\begin{aligned} v &= \frac{BY}{R}, \\ v' &= \frac{AV}{r}. \end{aligned}\right\} \therefore \frac{v}{v'} = \frac{BY}{AV} \times \frac{r}{R} = \frac{HB}{HA} \times \frac{r}{R}.$$

Or, since $HA = x$, and $HB = x + l$, $\frac{v}{v'} = \frac{r(x+l)}{Rx}$; whence, substituting in this equation the value of x, already found, and reducing, we have finally

$$\frac{v}{v'} = \frac{\pm r\sqrt{Rlr(R-l+r)} + Rlr}{\pm R\sqrt{Rlr(R-l+r)} - Rlr}.$$

This expression, with the assigned lengths of the link and levers, gives the values,

$$\frac{v}{v'} = \frac{77}{50} \text{ (very nearly), for Fig. 40.}$$

$$\frac{v}{v'} = \frac{13}{200} \quad \text{“} \quad \text{“} \quad \text{“ Fig. 41.}$$

51. Now a question of at least theoretic interest arises in regard to the action of the mechanism shown in Fig. 39. It is usually held, I believe, that only the component in the line of the link is effective in transmitting either force or motion from one lever to another. And, so long as there *is* any such component, there is no occasion, and, it may be said, no ground, for questioning the correctness of that dictum. Nor yet in the case of a single dead centre; thus, in Fig 39, the link AB merely rotates about its instantaneous axis B; which is at the instant stationary, for the simple reason that it can not move both ways at once: so, although there is no component in the line of the link, no motion nor force is transmitted to the lever *CBE*.

But we may imagine BD in Fig. 40 to drive, with a constant velocity, for a limited distance on each side of the critical position; in that case the velocity of AC will be variable, of course, but it will not vanish at the instant of collapse, notwithstanding the fact that there will then be no component in the line of the link—for it has been shown that it must have a definite value as compared with that of BD. This arrangement, then, seems to present a remarkable exception to the general proposition referred to, whether regarded kinematically or dynamically.

52. It may be (and has been) said that owing to the elasticity of materials, to the impossibility of securing absolute freedom from "play" or lost motion,—in short, to the perverse nature of inanimate things and to all that prevents the realization of abstract desiderata, there will practically be a small rotation about A at the critical instant; after which there *would* be a longitudinal component of motion or of force. Were this the true explanation, this component would be very small in comparison with its effective lever arm, unless the rotation about A were very considerable; from which it would follow that the levers would "go hard" across the line of centres; and that the more, the better the fitting. So that, were the fitting absolutely perfect, the levers could only cross the line of centres by the aid of momentum, even if friction were entirely eliminated; and if once stopped upon that line, no force applied to either lever could move the other.

53. But a study of the action of a model made for the Stevens Institute of Technology leads me to the conclusion that the exact contrary is the case. The proportions of this model are the same as those of the figures accompanying this article; and the levers pass the critical position with the

greatest ease. From which it would seem that the link at that instant acts as a lever whose fulcrum is the instantaneous axis, and operates by side pressure instead of by end-long thrust or pull.

Another interesting point in regard to the action of the model is this: that when the levers are placed on the line of centres, the mechanism exhibits no hesitation in making a choice between the two possible positions of the instantaneous axis. If the lever *BD* is the driver, the combination moves as indicated by the diagram in Fig. 40; but when *AC* is the driver, the motions correspond to Fig. 41.

It may be added, in conclusion, that the dead points may also occur simultaneously when the two levers point in the same direction; and since the proportions may be varied in many ways, the diagrams may present appearances very unlike those here given. But this method of reasoning may be applied in all cases; and in no one of them will the velocity ratio be found indeterminate.

"SLOW ADVANCE AND QUICK RETURN" MOTION PRODUCED BY ELLIPTICAL WHEELS.

54. In Fig. 42, *C* and *D* are the fixed centres of a pair of elliptical wheels, in contact at *P*. In order that these may move in continuous rolling contact, the two ellipses must be identical, each must be centred upon one of its foci, and the distance between the fixed centres must be equal to the major axis; in practice, both wheels must be provided with teeth, which are omitted in the diagram, since the action is best seen and discussed by consideration of the pitch curves only.

Since the point of contact must always lie upon the line of centres *CD*, its motion must be perpendicular to that line,

and its linear velocity is the same whether it be regarded as belonging to one ellipse or the other, so that the angular velocities are inversely proportional to the contact radii; thus, letting

v = ang. vel. about D, and
v' = " " " C, we have
$\frac{v}{v'} = \frac{PC}{PD}$, in the present position,

and setting off the equal elliptical arcs PEK, PIJ, then, when J and K meet, we shall have

$\frac{v}{v'} = \frac{JC}{KD}$; or, drawing JC and joining K with F, the free focus of the right-hand ellipse, this latter value may be written

$$\frac{v}{v'} = \frac{KF}{KD}.$$

In order to bring K and J into coincidence, it is apparent that while the driver turns as indicated by the arrow, through the angle PDK, the follower must turn in the opposite direction through the angle PCJ; and this, it is to be noted, is equal to the angle PFK.

55. These ellipses are so situated and proportioned that PC and PF are respectively perpendicular to the major axes; and, producing these lines to cut the ellipses in H and G, the arcs PIH and PEG are equal; whence it follows that while the driver turns through the angle GDP, the follower will have made a half revolution.

While it is making the other half, the driver must complete its own, thus turning through an angle much greater

than GDP; since the two ellipses have equal perimeters and necessarily revolve in equal times about their fixed centres of rotation.

Now let a pin, fixed in the follower at any point A in the major axis, actuate a link whose farther extremity B is constrained by guides (not shown) to travel in the line of that axis, then the traverse of B along that line will be equal to $AL = 2AC$; it will make the upward stroke while the driver turns through the angle GDP, and the downward stroke while the driver is completing its revolution. If, then, the driver turns uniformly, the down stroke of B will occupy a greater time than the up stroke; which is just what is required in a shaping machine, which calls for a "slow advance" of the tool-holder while making the cut, and a "quick return" for saving time while no work is done. As here arranged, the cut would be made on the down stroke, and by the "pull" of the link; this is merely for the sake of saving space in the illustrative diagram; and it will be evident that had B been below A instead of above, the condition would have been reversed, the work being done by the "push" of the link; both arrangements have been used in practice, but the kinematic action is the same whichever be adopted.

56. Now if, in the design of a shaping machine to be actuated by elliptical wheels as above shown, the ratio of the times to be occupied by the advance and the return be assigned, the first thing is to determine the eccentricity of the ellipses which will satisfy the requirement: the manner of doing this may also be illustrated by Fig. 42. Let the time of the return be to the time of the advance as m is to n, for instance; then about any point D on the indefinite horizontal line HN describe a circle with any radius MD.

Divide the upper semicircumference into a number of equal parts represented by $m + n$, and set off MF equal to m parts, then $FN = n$ parts. Draw DF and produce it; this will be the direction of the major axis of the driver: draw at F a perpendicular to DF, cutting HN in P; then P is a point upon the ellipse, of which D and F are the foci, and the major axis is equal to $PF + PD$. On HN set off PC equal to PF; produce FP to A, making PA equal to PD; then C and A will be the foci of the other ellipse. In the figure, the advance is assumed to occupy three times as many seconds as the return; that is to say, $m = 1$, $n = 3$, consequently the semicircumference MFN is divided into 4 equal parts, and MF being made equal to one of these parts, FN is equal to the other three.

It need hardly be added that the link may be operated by a crank keyed in any convenient position upon the shaft of the driven ellipse, with the understanding that it must be upon a dead centre when the ellipses occupy the relative positions shown in Fig. 42.

57. Our problem now is, to construct a diagram representing the varying velocity of B throughout the revolution of the driving ellipse, whose angular velocity is assumed to be constant. In order to do this, we must be able to determine at any given instant the motion of the crank-pin A (which, for the sake of simplicity only, has been placed in the focus of the driven ellipse). The *direction* of A's motion is known, being always perpendicular to AC, which line revolves about C in a direction opposite to that of the driver. In determining its *velocity*, we have only to remember that angular velocity may be represented by the linear velocity of a point at unit distance from the axis (the magnitude of this unit, for purposes of comparison, being arbitrary); and

FIG. 43.

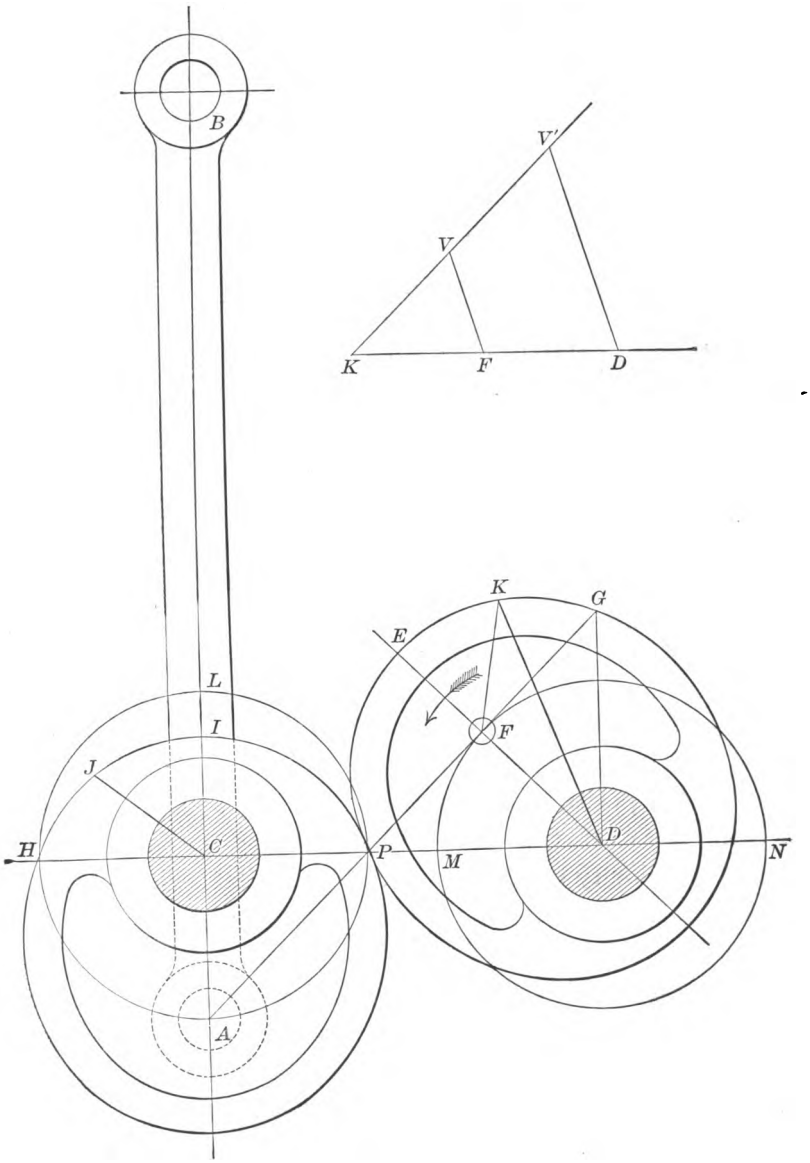


FIG. 42.

we have already shown that the angular velocities of the driver and follower are inversely proportional to the radii of contact, at any given instant.

Assuming then, in Fig. 42, CA, which is equal to DF, as the unit distance, and assigning to F any linear velocity at pleasure; from what precedes we shall have, when for instance J and K meet upon the line of centres,

$$KF : KD :: \text{lin. vel. } F : \text{lin. vel. } A.$$

The required velocity of A may then be easily determined graphically as in Fig. 43. Upon either of two lines intersecting in K at any convenient angle, set off from K the distances KF, KD, taken from Fig. 42; on the other line set off KV, the assigned linear velocity of F; draw VF, and through D a parallel to it, cutting the prolongation of KV in V'; then KV' is the required linear velocity of A, since by similar triangles we have

$$KF : KD :: KV : KV'.$$

58. We repeat in Fig. 44 the construction for determining the velocity of B when that of A is known; AC representing the position of the crank, AB that of the link, and CB the line of travel. Let AM, perpendicular to AC, represent the velocity of A; this has a component Am in the line of the link, to which BN must be equal. Then drawing at N a perpendicular to AB, cutting CB in R, we have BR as the resultant velocity of B. This work may be abbreviated by drawing AR' parallel to CB, and limited by the prolongation of Mm perpendicular to AB, since the triangle AmR' thus formed, is similar and equal to BNR.

Now, in constructing a velocity diagram such as is required, it is clearly advisable not to select random points

or contact between the driver and follower, like K and J in the preceding illustrations, but to proceed in an orderly and

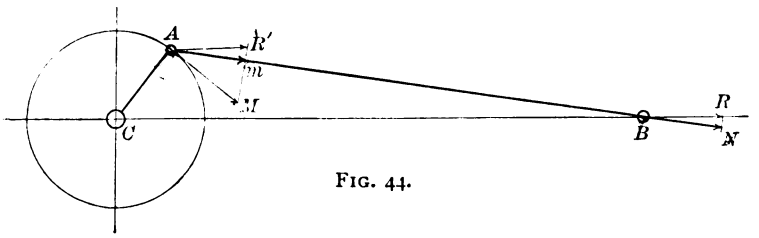


FIG. 44.

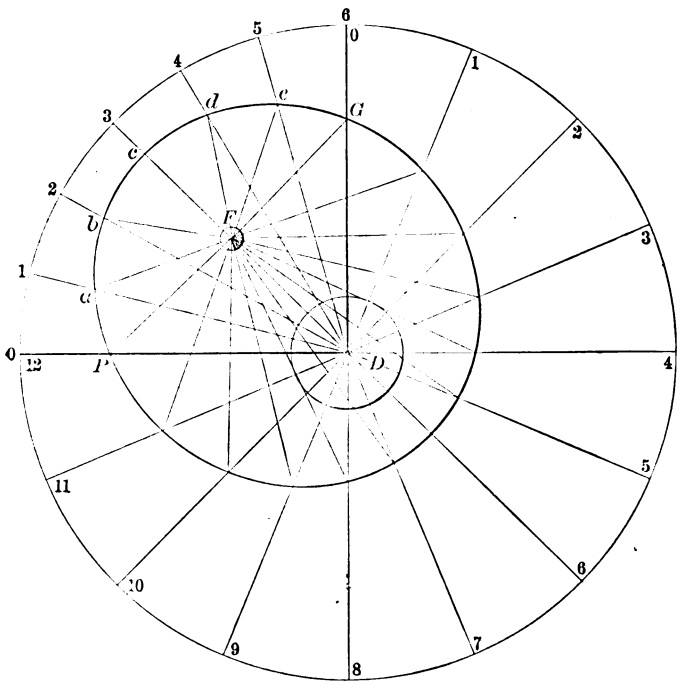


FIG. 45.

systematic manner. For the purpose of showing clearly how this may be done, we have in Fig. 45 reproduced the driving ellipse in the same position as in Fig. 42; *PD* being

the initial contact radius, and *PDG* the angle through which the driver turns during the "quick return" stroke of *B*. About the fixed focus *D*, describe a "measuring circle" large enough to lie well outside the ellipse, as in the figure, and produce *DP*, *DG*, to cut its circumference. Divide the quadrant which measures the angle *PDG* into any number of equal parts, and from the points of division, 1, 2, 3, etc., draw radii to *D*, cutting the perimeter of the ellipse at *a*, *b*, *c*, etc.; from these points draw finally the lines *aF*, *bF*, etc., to the other focus of the ellipse.

The points *a*, *b*, etc., will evidently become contact points at equal intervals of time; and as each one does so, the corresponding positions and velocity of the crank-pin *A* are to be determined as above explained in reference to the point *K* in Figs. 42 and 43, and finally the velocity of *B* for each position is to be found as in Fig. 44.

59. These values of the velocities of *B* are the ordinates of that portion *XUY* of the curve in Fig. 46, which pertains to the return stroke. Any distance *XY* on the line of abscissas, representing the time occupied by that stroke, is divided into as many equal parts as the quadrant in Fig. 45; any point of subdivision, as *b* for instance, represents the instant when the corresponding point of the ellipse in the preceding figure becomes a contact point, and the ordinate represents the velocity of *B* at that instant.

Since the advance stroke is made during the remaining three-fourths of the revolution, that fraction of the measuring circle is next divided into any convenient number of equal parts as indicated by the figures 1, 2, 3, etc., inside the circumference in Fig. 45; in Fig. 44, $YZ = 3XY$, is similarly subdivided, and the diagram is completed by setting up at each point an ordinate representing the velocity of *B* at the

instant represented by that point, ascertained by constructions identical with those already explained.

Now, since the abscissas represent times and the ordinates represent velocities, and since also

$$\text{Time} \times \text{Velocity} = \text{Space},$$

it is clear that the areas of the two curves *XUY*, *YWZ*, should be equal, each representing the length of the stroke.

60. By the mode of construction, as seen in Fig. 42, the major axis of the driving ellipse bisects the angles through which it turns during both strokes. And it is practically advantageous to subdivide both these angles so that, as in Fig. 45, the successive contact points upon that ellipse, which are made use of in determining the ordinates of the velocity diagram, shall be symmetrically disposed with respect to that axis. Because, the ratio of the focal distances, and therefore the velocity of the crank-pin *A*, will be the same for *a* and *e*, for *b* and *d*, and so on, thus saving much time in determining the velocities of *A*. In regard to the number of ordinates to be determined, it is impossible to give any specific directions; but, in constructing these or any other curves, it is plain that the points through which they must pass should be nearest each other where the curvature changes most rapidly; as, for instance, in the region of *U* the vertex of the return stroke-curve in Fig. 46.

61. Another diagram of interest is given in Fig. 47, where the curve *UVW* represents the varying *angular* velocity of the driven ellipse, the constant velocity of the driver being represented by the horizontal line *MN*. Since the driver and the follower turn in opposite directions, it would, perhaps, be more consistent to place *UVW* and *MN* on opposite sides of *AB*; but the directional relation in

a case like this is hardly likely to be lost sight of, and the varying velocity can be more readily compared with the constant one by adopting the arrangement shown in the figure.

The ordinates of this curve are the linear velocities of

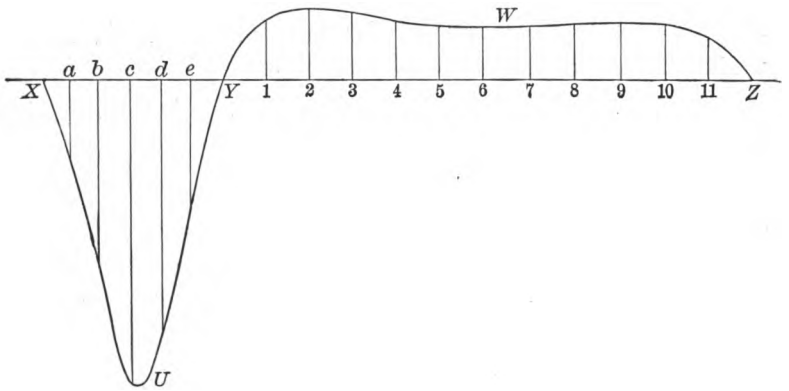


FIG. 46.

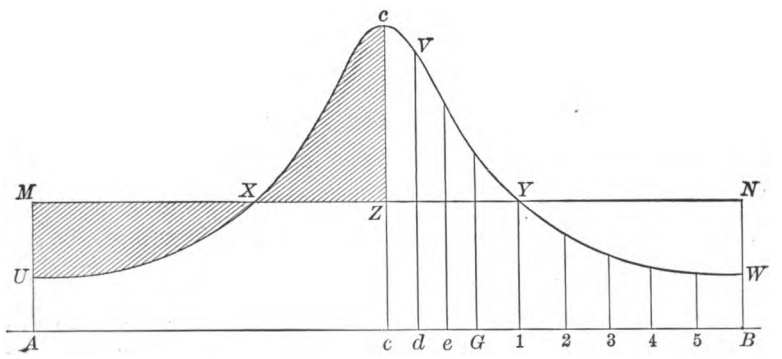


FIG. 47.

B, as determined in constructing the preceding diagram. The velocity of A will be greatest when E of Fig. 42 (corresponding to c of Fig. 45) is the driving point of contact, and least at the end of a half revolution, when the

opposite ends of the major axes come together. In the former position we shall have for the velocity ratio, $\frac{v'}{v} = \frac{ED}{EF}$, and in the latter $\frac{v'}{v} = \frac{EF}{ED}$, the one value being the reciprocal of the other; and if either of these positions be selected as the initial one, it is clear that the resulting diagram will be symmetrical with respect to its central ordinate, because the ellipse is symmetrically divided by its major axis.

We prefer to make the first ordinate the least, as AU in Fig. 47, where AM represents the constant velocity of the free focus of the driver, and is equal to KV in Fig. 43. Since each ellipse completes its revolution in the same time, it follows that the area of the curve UVW must be equal to that of the rectangle AN, and that the areas of the shaded portions UMX, XZc, must be equal to each other.

62. It may be added that in using elliptical wheels for this purpose, the writer hereof has shown that any given ratio between the times of the advance and the return may be secured by means of ellipses whose eccentricity varies between certain limits, so that a given pair of elliptical wheels might be used for shaping machines with different conditions as to the relative times of advance and return: in the course of the investigation it was also demonstrated that for a given ratio between those times, the construction illustrated in Fig. 42 gives the *minimum eccentricity*.

This is a point of practical importance; for since it is clearly impossible to preserve a running balance in the rotating parts of mechanism of this description, it is as clearly advantageous to reduce the unavoidable vibration to the lowest attainable limit.

OTHER "SLOW ADVANCE AND QUICK RETURN" MOTIONS.

63. In Fig. 48 is represented the device known as "the crank and slotted arm," which in various forms has been extensively used for producing the slow advance and quick return motion in shaping machines. D is the fixed centre of the driving crank, whose pin P turns in a block which slides freely in the slotted arm CA, of which C is the fixed centre. To the extremity A of this arm is jointed a link, of which the other extremity B is constrained by guides (not shown) to travel in the line KH. The excursion of the arm CA is plainly limited by the lines CG, CI, tangent at E and F to the circular path of P.

With G and I as centres, describe with radius AB arcs cutting the line of travel of B in H and K; these points limit the traverse of B in its assigned path. When P is in the position shown, let PO perpendicular to PD represent its linear velocity.

Resolve PO into components PV in the line AC, and PL perpendicular to it; the former represents the sliding, which is accommodated by the slot, and the latter, PL, is the linear velocity of the point P in the line CA, in rotation about C. Therefore, AM, perpendicular to AC, and limited by the prolongation of CL, is the linear velocity of A. The component of AM in the direction BA is AN, to which BQ must be equal, and a perpendicular to BA at Q cuts the line KH in R, giving BR as the resultant velocity of B.

As in previous cases, this may be abbreviated by drawing through A a parallel to KH, cutting MN produced in T; since the triangle ANT thus formed is similar and equal to BQR.

64. In this arrangement it is obvious that the arm CA

FIG. 48.

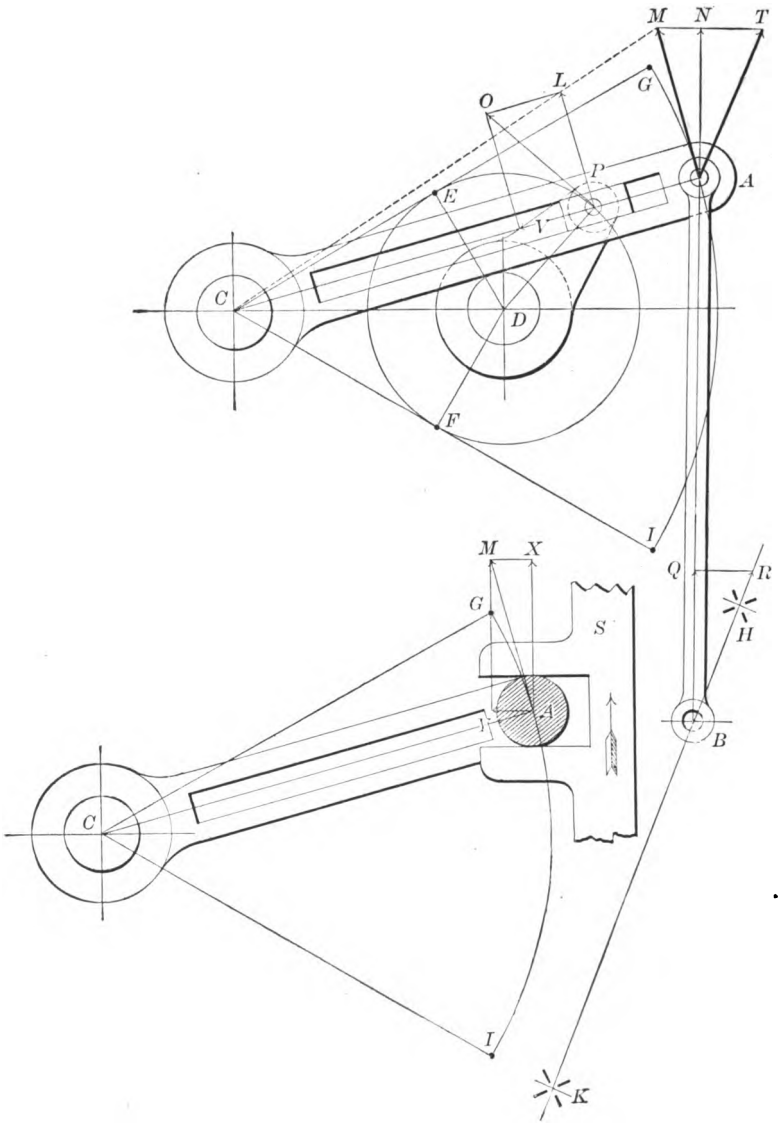


FIG. 48*a*

will swing through the arc IG while the driving crank turns through the arc FPE, and A will return from G to I while the crank turns through the remaining part of the circumference, EF.

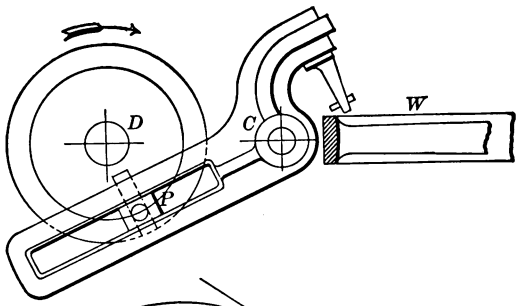
Consequently, if the ratio between the times of the advance and return be assigned, as for instance as m is to n; describe a circle about any centre D with any radius DP, divide its circumference into $m + n$ parts, and make EF equal to n parts; then $EPF = m$ parts. At E and F thus determined, draw tangents to the path of P, intersecting at C: this determines the proportions of the moving pieces and the relative positions of the fixed centres—it being obvious that the lengths of CA and of AB, as well as the direction of B's travel, are entirely arbitrary in theory, although in practice they are limited by various considerations which need not here be discussed.

The link, indeed, may be dispensed with altogether; thus in Fig. 48*a*, the outer extremity A of the slotted lever is formed into a pin projecting from the front side of the lever; this pin works between jaws formed on a piece S, by which the cutting tool is carried, and this piece is constrained by guides (not shown) to travel in the direction of the arrow. In this case, the motion AM of the point A is to be resolved into the components AX, and AY; the former alone is effective, and represents the resultant motion of the sliding head S.

65. This combination has also been used in a curvilinear slotting machine, the tool being carried directly by a curved extension of the vibrating arm, as shown in Fig. 49. This figure represents an application to the work of reducing the thickness of portions of the rim of a locomotive driving-wheel W, when the latter had proved not to be in correct

running balance: as described and illustrated in Professor Goodeve's "Elements of Mechanism."

FIG. 49.



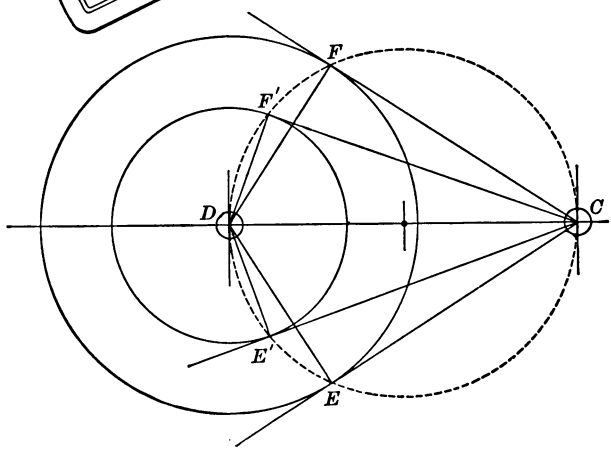


FIG. 49*a*.

The crank is here replaced by a disk, provided with a slot to enable the pin *P* to be fixed at a greater or less distance from the centre *D*, in order to regulate the length of the stroke. This adjustment, however, affects the ratio between the times of the advance and return; as the effective crank-arm is diminished, the relative time of the return will be increased. This will be clear by the aid of Fig. 47*a*,

where it is obvious that the angle DCE' is less than the angle DCE, and since the angles at E' and E are both right angles, it follows that CDE' is greater than CDE.

66. In Fig. 50 is shown the modification of the "crank and slotted arm" device known as the Whitworth motion; it differs from the preceding in that the centre C lies within the path of P instead of outside of it, so that CA makes a complete rotation for each revolution of the driving crank. The path of P is divided at E and F, exactly as in the preceding case, for the purpose of securing any desired ratio between the times of the advance and return; the chord EF is then bisected in order to locate C; and this chord is also the line of travel, in the practical use of the movement. In determining the speed of B when that of P is assigned, the proceeding is precisely the same as in Fig. 48, and since the two diagrams are lettered similarly throughout, no further explanation is required.

67. It is very evident that the device shown in Fig. 50 is in that form perfectly impracticable as a working machine, since the shaft C would interfere with the crank DP, if it projected behind CA, while if it projected in front of the slotted arm it would interfere with the link AB. These difficulties were most ingeniously evaded by the distinguished inventor, and an extremely serviceable machine constructed in the manner illustrated in Fig. 51. The crank is here replaced by a wheel W, riding loose upon a fixed shaft whose centre is D, and driven by a pinion w; in the front face of this wheel is fixed the driving crank-pin P. The fixed shaft D has bored in it a hole at C, as shown more clearly in Fig. 52. Into this hole projects a pin forming a part of the piece SS, Fig. 51, at the back of which is a slot in which slides a block fitted upon the pin P. In the front

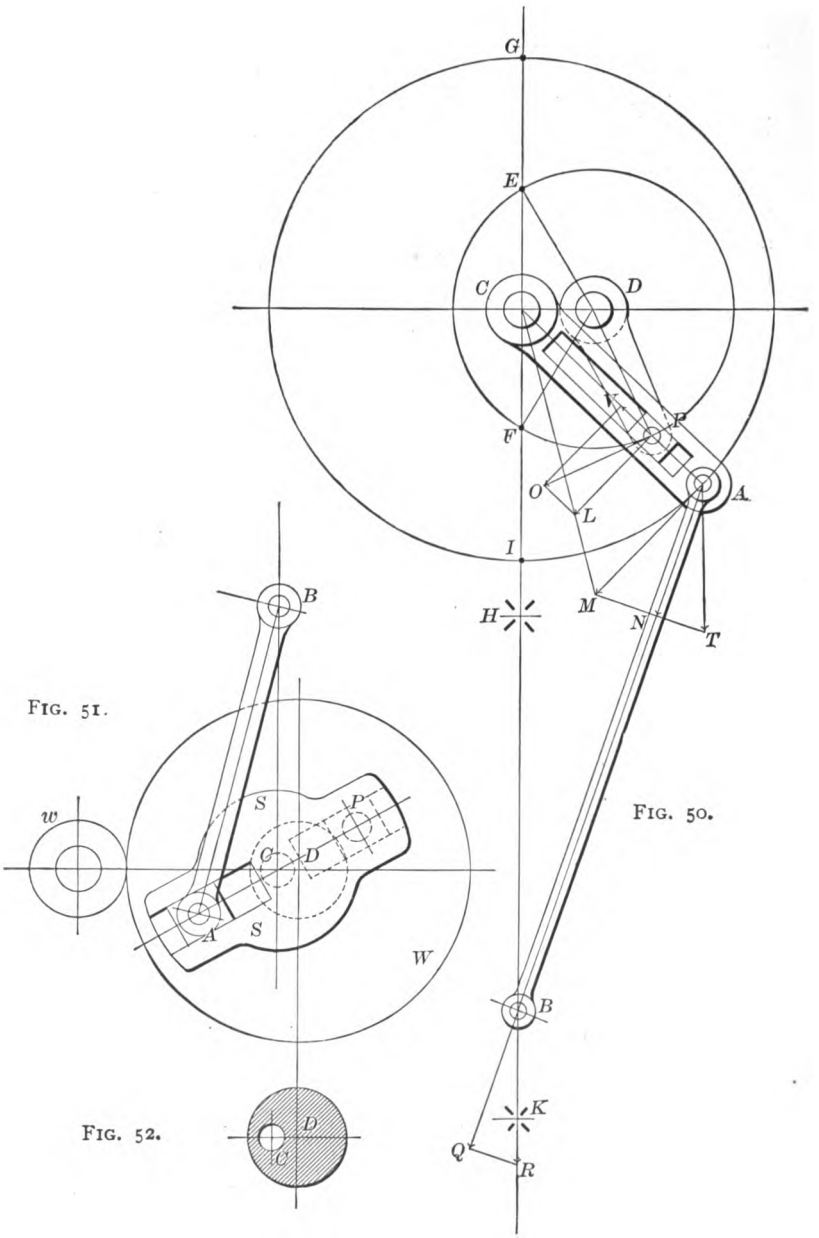


FIG. 51.

FIG. 50.

FIG. 52.

face of *SS* there is also a slot in which the pin which drives the link *AB* can be fixed in any desired position.

By varying the distance between *A* and *C*, then, the length of the stroke may be controlled at pleasure; but in this case there is no change in the ratio between the times of the advance and the return, since the relative positions of *P*, *D*, and *C*, which fix this ratio, are not affected by any variation of *AC*.

68. Another method of producing a slow advance and quick return movement, by the use of levers and links only, is shown in Fig. 53. In this combination, the driving crank *DA*, by means of the link *AB*, actuates the vibrating lever *BC*. To the extremity, *B*, of this lever is pivoted the link *BP*, whose free end, *P*, is constrained to move to and fro in the path *HK*.

Supposing as before that the time of traverse in one direction along *HK* is to be to the time occupied in the return as *m* is to *n*, the skeleton movement is laid out thus: about any centre *D* describe a circle with any assumed radius, as *DA*; and divide the circumference at *E* and *F* so that $EAF = m$ parts, while $FE = n$ parts, as in the preceding cases. Assuming any reasonable length for the first link *AB*, set off *EDI* equal to this assumed length; this locates the point *I*, the crank being at an inward dead point. Next draw *DF* and produce it, making *FG* also equal to *AB*; then *G* is the extremity of this link when the crank is at an outward dead point. Draw *IG*, bisect it by a perpendicular, and on the bisector take any point *C* as the fixed centre of the vibrating lever *CB*; in practice the angle *ICG* should never exceed 60°, and if possible it should be less: the length of *BP* and the line of travel of *P*, are clearly arbitrary.

This being merely a combination of links and levers, with no new or peculiar features, it is not necessary to give a detailed explanation of the process of determining the velocity of P when that of A is given.

69. But it may be suggested that in constructing a velocity diagram for either of these combinations, the arc EF should be subdivided into equal parts, and the remainder of the circumference then divided in like manner by

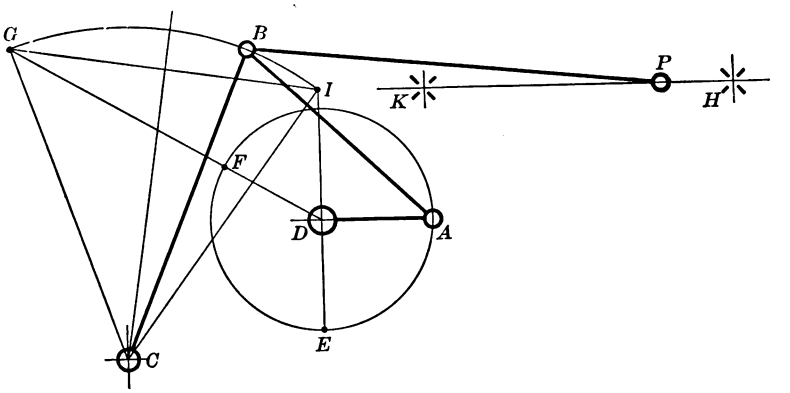


FIG. 53.

itself; it might at first glance appear more expeditious to divide the whole circumference at once—but the proceeding suggested will, on trial, be found far more satisfactory.

It is to be admitted that the combination in Fig. 53 is not particularly suitable for use in a shaping machine; but it is given as one of various methods of producing the required motion,—and moreover, combinations essentially the same in principle have been practically employed in various forms of hot air and gas engines, with very satisfactory results.

70. Thus far in determining the motion of a piece driven by a lever through the intervention of a connecting-rod. or

link, we have adhered to the direct process, by means of the longitudinal component of the motion of the driving point; because the reasoning is clear and simple, and perfect familiarity with that method is essential. We now proceed to explain one or two other methods, which in some cases are more expeditious, although the reason why they are correct is not by any means so obvious.

Fig. 54 shows the crank and connecting-rod of a direct-acting steam-engine, the cross-head pin O travelling in the horizontal line OC. At C erect the vertical line CX, and produce the line of the connecting-rod OP, if necessary, to cut CX in the point A. Then, if the velocity of P is represented by a line equal to CP, the velocity of O will be represented by a line equal to CA.

For, setting off PM perpendicular and equal to PC, and drawing MB perpendicular to PA, and PB perpendicular to CA, the triangle PAC is similar and equal to the triangle PMB. And it has already been shown that if PM is the velocity of P, PB is equal to the velocity of O; and CA is equal to PB.

If for any reason it is desirable to represent the velocity of the crank-pin by a line greater or less than CP, as for instance CL, it will now be obvious that, drawing LK parallel to OP, we shall have CK equal to the velocity of the cross-head pin.

71. Fig. 55 also represents the crank and connecting-rod of a direct-acting engine. In this construction, PR, the velocity of the crank-pin, of any magnitude at pleasure, is set off on the prolongation of CP; draw through R a parallel to PO, cutting at T a vertical line through O, then OT is equal to the velocity of O. For, first setting off as before PM perpendicular to PC and equal to PR, draw PQ parallel

and equal to OT, MB perpendicular to RQ, and PB perpendicular to PQ. Then the triangles PMB, PRQ, are similar and equal, and we have $PQ = OT, = PB$.

If we lay off the velocity CL from the centre C, and

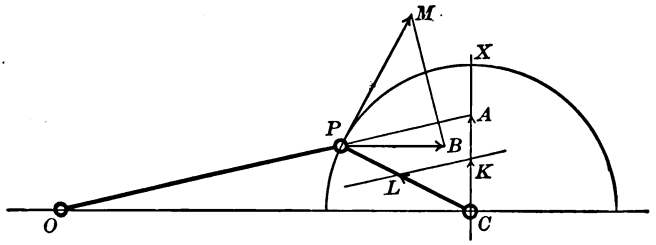


FIG. 54.

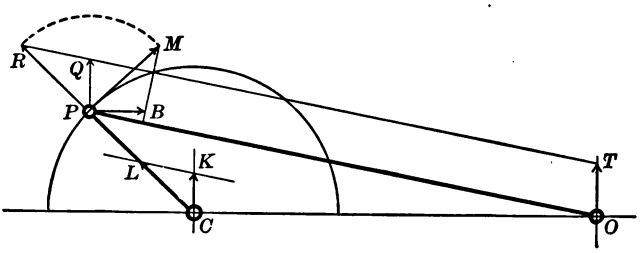


FIG. 55.

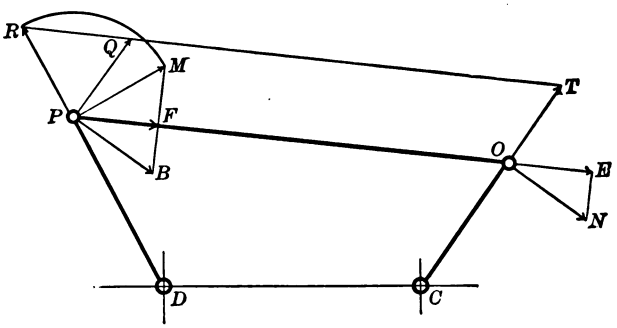


FIG. 56.

draw LK parallel to PO, cutting the vertical through C in K, the triangles CLK, PRQ, are similar, so that CK will be the velocity of O, as in Fig. 54.

72. In Fig. 56 we have two levers *DP*, *CO*, connected by a link *PO*. The construction here is analogous to that of the preceding figure; setting off *PR*, the assigned velocity of *P*, on the prolongation of *DP*, draw through *R* a parallel to *PO*, cutting the prolongation of *CO* in the point *T*: then *OT* is the velocity of *O*. And the demonstration is also the same as before; for drawing $PM, = PR$, perpendicular to *DP*, draw also *PQ* equal and parallel to *OT*, *MB* perpendicular to *RQ*, and *PB* perpendicular to *PQ*; we have then, by reason of the two similar and equal triangles, $PQ = OT, = PB$.

In order to save the reader the annoyance of referring to preceding diagrams, we have in Fig. 56 repeated all the steps of the direct determination; thus, *PF* is the absolute longitudinal component of *PM*, and *OE* is equal to *PF*; *ON*, the resultant motion of *O*, is perpendicular to *CO*, and is limited by drawing *EN* perpendicular to *PO*. Then, the triangles *OEN*, *PFB*, being similar and equal, we have $PB = ON$.

And thus we see that the simplest demonstration that the expeditive processes are correct, is made by proving that they give results concordant with those of the original direct method.

73. We have already explained and illustrated the representation of varying *angular* velocity, in the case of a crank rotating continuously in the same direction. That is no doubt the case in which such a representation is most striking and most readily understood; but the same methods of construction are equally applicable in the case of the circular reciprocation of a vibrating lever, although the result does not appeal so directly to the eye, and the diagram is not quite so easily read.

To illustrate, we give in Fig. 57 a skeleton of the "crank and slotted arm" movement, like that in Fig. 48; and in Fig. 58, a time-diagram of the angular velocity of the vibrating arm AC. The slotted arm AC moves from the position CE to the position CF while the pin P of the driving crank travels through the arc EZF, and back again while the crank-pin describes the remaining arc of the circumference, FE; in this instance the former arc is twice the latter,—that is, if used in a shaping machine, the time of the advance will be twice that of the return: consequently, in Fig. 58 we make

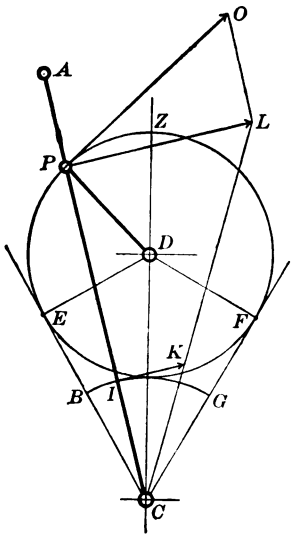


FIG. 57.

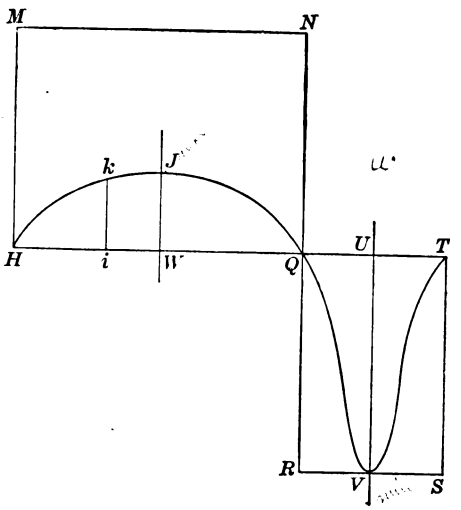


FIG. 58.

$HQ = 2QT$. In Fig. 57 is shown the process of determining one ordinate of the required curve. Since angular velocity = linear velocity of a point at unit distance from the centre, the first step is to describe about C an arc BG with radius $CB = DE$. In this particular case this arc will be tangent to the arc EF, because $EDC = 60°$, whence $CD = 2DE$.

74. Now, when the driving crank is in any position as DP, assign to P any velocity as PO; the component PL perpendicular to PC is the velocity of the point P on PC. Then, PC cuts the arc BG at I; and IK, perpendicular to PC and limited by LC, is the value of the ordinate sought. Then, in Fig. 58, divide HQ so that

$$Hi : iQ :: \text{arc } EP : \text{arc } PZF,$$

and set up the ordinate ik equal to IK.

Now, making $HM = PO =$ constant velocity P, the rectangle HN represents the space traversed by the crank in the time HQ, which is the arc EZF; and the area of the curve HJQ represents the space traversed by the point B in the same time, or the arc BG, $= \frac{1}{4} EZF$: therefore the area of the rectangle should be four times that of the curve. On the return stroke, the ordinates are negative, QR is equal to PO, and since $GB = \frac{1}{2} FE$, the area of the curve QVT is half that of the rectangle QS.

It is hardly necessary to point out that the curves HJQ, QVT, are symmetrical about their central ordinates JW, UV, so that it is only necessary to determine the ordinates for one half of each.

75. At the outset, a "velocity diagram" was defined as a curve whose abscissas represent *times*, while the ordinates represent the velocities, linear or angular, of a moving point at the instants indicated by the feet of the ordinates.

This is what is distinctively called a "**velocity-time**" diagram, and attention has purposely been thus far confined to it alone, because we consider it to be the most explanatory, and most generally useful, method of graphically exhibiting the phenomena of varied motion: but it is not by any means the only method.

In Fig. 59, we have shown a skeleton of the "Whitworth Motion" (Fig. 50); *DP* is the driving crank, *CA* the slotted arm, *AB* the link, *BC* the line of travel. Assign to *P* any velocity at pleasure, determine from that the linear velocity of *B*, and set down, perpendicular to the line of travel, the ordinate *B2'* equal to that velocity. Do the same for as many different positions of the crank and arm as may be deemed necessary, and draw a curve through the extremities of the ordinates, for both the advance and the return. This curve is technically called a **velocity-space** diagram, and the ordinates of course show the velocity of *B* in its different positions.

These same ordinates, it is clear, are the ones used in constructing the "velocity-time" diagram, Fig. 60: in the former the abscissas represent varying distances traversed, in the latter they represent the equal times of traversing them.

76. As a matter of interest, we give in Fig. 61 the angular velocity-time diagram for the same movement; in which *AM* is the constant angular velocity of *P*, and the curve *UVW* represents the varying angular velocity of *A*.

It is obvious from inspection of Fig. 59 that the minimum velocity *AU* will be reached when both the crank and the slotted arm are on the vertical line and pointing upward; and the maximum, *ZV*, after a half revolution from that position, when both point downward. The ordinates of the curve are numbered to correspond with the subdivisions of the circles of Fig. 59; and it is to be noted that the curve crosses the horizontal line *MN* at the point *X*, coincident with the ordinate 8–8, at which instant the slotted arm is horizontal.

The ratio of the time of advance to the time of return

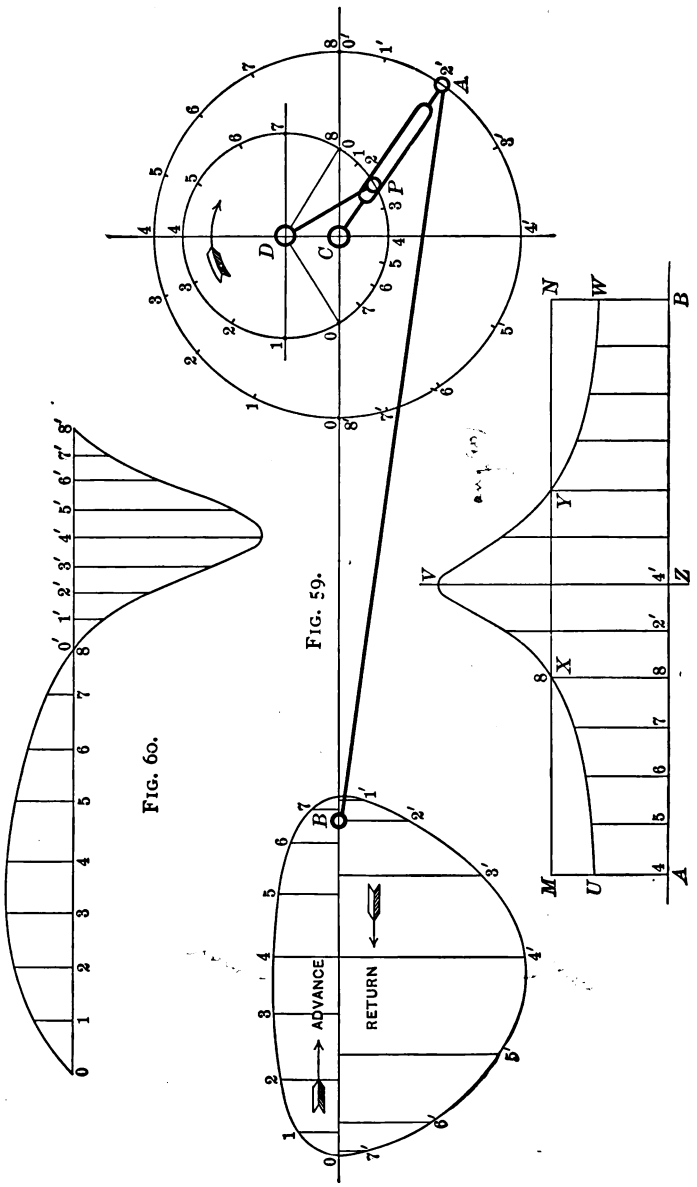


FIG. 59.

FIG. 60.

FIG. 61.

is, in this particular case, two to one; but it is of interest to observe that the above coincidence will occur, whatever the ratio. This may be demonstrated by the aid of Fig. 62, where PB is the assigned velocity of P, PF the component effective in driving CA, and AR the resultant velocity of A. We have then,

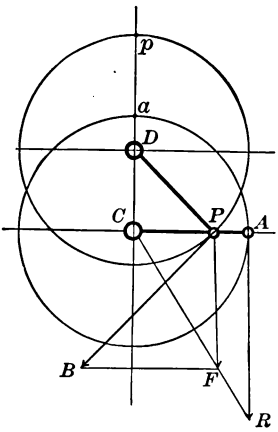


FIG. 62.

$$PF : PC :: PB : PD, = CA,$$
$$\therefore PF \cdot CA = PC \cdot PB;$$

also,

$$PC : CA :: PF : AR,$$
$$\therefore PC \cdot AR = PF \cdot CA, = PC \cdot PB,$$

whence, $AR = PB$.... ..... Q. E. D.

77. Varying velocity may also be represented by a curve drawn through the extremities of lines of different lengths radiating from one central point, forming what is called a *polar* velocity diagram. Thus in Fig. 63, the circle represents the path of the pin A in Fig. 59; its circumference is similarly subdivided, and on the radii are set off the corresponding angular velocities, taken from Fig. 61. The resulting curve, then, serves to show the angular velocity of the slotted arm in any given position.

In Fig. 64, the circle represents also the path of A in Fig. 59, but it is divided into equal parts like the path of P in that figure. The distances set off on the radii are the same as in Fig. 63, and the polar diagram exhibits the angular velocities of the slotted arm at equal intervals of time. And it is quite obvious that similar diagrams could be made to represent the *linear* velocities of A at different points, or at different times.

78. Fig. 65 represents the crank, link, and vibrating lever movement that was given in Fig. 53 as one means of pro-

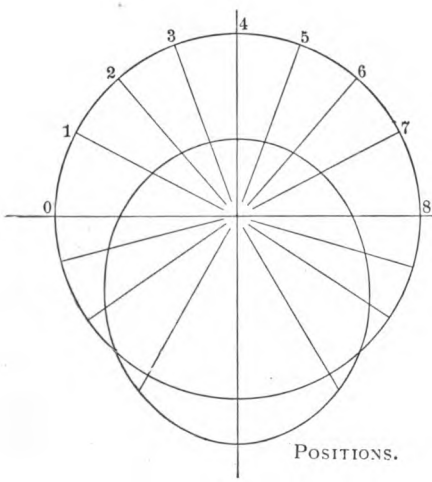


FIG. 63.

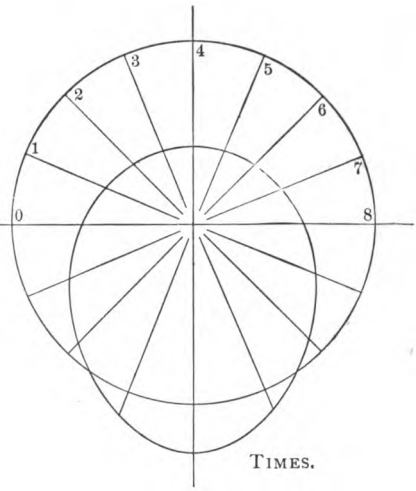


FIG. 64.

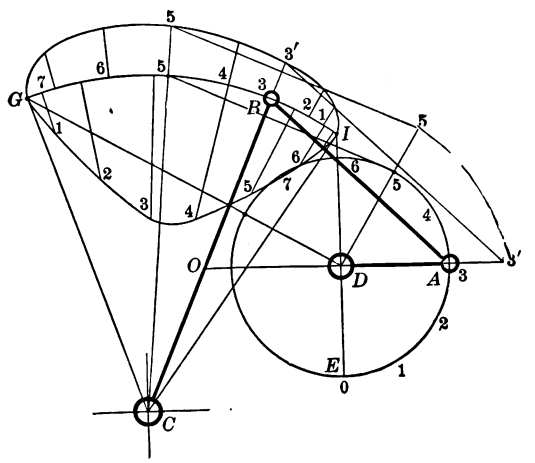


FIG. 65.

ducing a slow advance and quick return motion. The linear velocities of B in its different positions are laid off from the

arc *IG* upon lines radiating from *C*, so that the resulting curve is in effect a modified velocity-space diagram in which the line of abscissas is a circular arc instead of a right line.

These velocities, as shown, are determined by the method of Fig. 56; and this figure incidentally shows another demonstration that the method is correct. For, when the parts are in the positions drawn in heavy lines, *O* is the instantaneous axis of the link; consequently, linear velocity *A* : linear velocity *B* :: *OA* : *OB*; and the triangles *OAB*, *O*3′3′, are similar.

79. Still another mode of representing varied motion is shown in Fig. 66, the movement being merely the crank, connecting-rod, and cross-head, of a direct-acting engine.

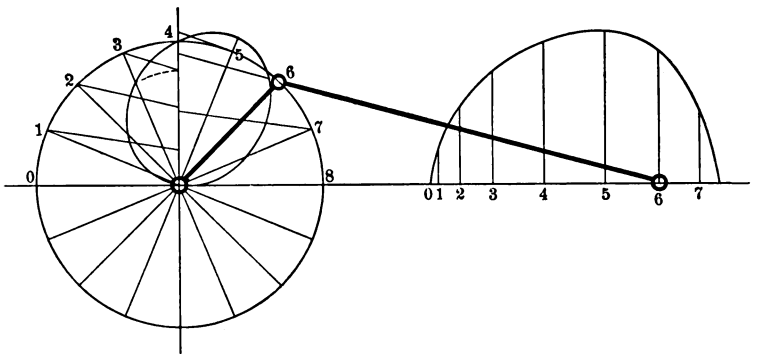


FIG. 66.

The circular path of the crank-pin is divided into equal parts at 1, 2, 3, etc., which represent equal intervals of time; and on the radii through these points are set out from the centre, the velocities of the cross-head pin at the corresponding instants: and the curve determined by these points on the radii is for some occult reason designated as a polar velocity-time diagram of the motion of the cross-head pin. The linear velocity of the crank-pin being here represented

by a line equal in length to the crank, the required velocities are, as is obvious on inspection, determined by the method of Fig. 54.

But it must be admitted that this diagram does not so instantly appeal to the eye, or explain its own message so clearly, as those do in which the radial ordinates, as in Figs. 63 and 64, indicate the velocities of the radii themselves in circular movement.

80. The utility of velocity diagrams is nowhere more conspicuous than in comparing the actions of different combinations which effect the same result.

For example, Fig. 67 is the skeleton of an oscillating engine, with the trunnions at the end of the cylinder; in Fig. 68 the trunnions are placed at the middle of the length of the cylinder; and Fig. 69 is the common crank and connecting-rod movement: the lengths of all the cranks are equal. Supposing the rotative velocity to be also the same, it is of interest to compare the piston speeds with each other, and also with the piston speed of another engine having an infinite connecting-rod, or slotted cross-head, of which no skeleton is given.

The velocity-time diagrams of these four arrangements, for one stroke from left to right, are given in Fig. 70. That for the engine with the infinite connecting-rod is the curve marked *A*; this, being a perfect sinusoid, is symmetrical about its central ordinate *a*, which is also the maximum, and equal to the constant velocity of the crank-pin; the crank-arm being at that instant vertical.

81. Curve No. 1 shows the piston speed of the oscillating engine of Fig. 67; the maximum ordinate *c*, is equal to *a*, but since this corresponds to that phase of the movement when the piston rod is tangent to the path of the crank-pin.

which does not occur until after the crank has passed the vertical line, this ordinate is some distance to the right of *a*.

The engine of Fig. 68 gives curve No. 2; this is quite similar to No. 1, but its maximum ordinate, *d*, is still further

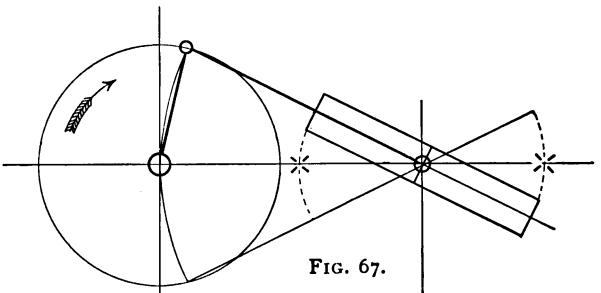


FIG. 67.

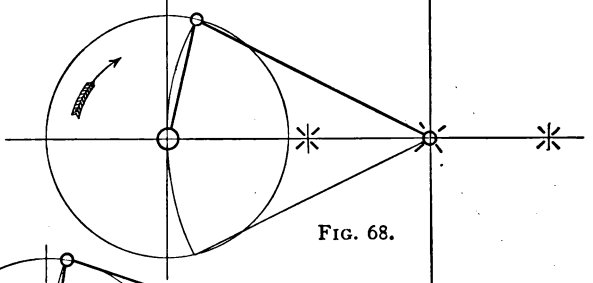


FIG. 68.

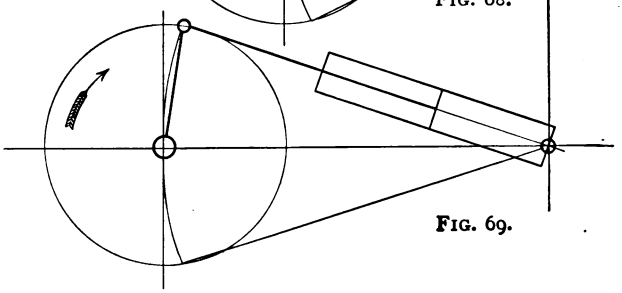


FIG. 69.

to the right—the reason for which is quite apparent from a comparison of the two movements.

The engine of Fig. 69 gives the curve No. 3, which is strikingly different from either of the others, in having *two*

ordinates, *a* and *b*, which are equal to the velocity of the crank-pin: the maximum ordinate lies somewhere between these two, but there is no geometric process known for determining its exact location. If, however, as in the figure we find by trial and error the centre *o* of a circle which sensibly agrees with the curve for a reasonable distance in the region of the vertex, a vertical ordinate *k* through the centre will, it is safe to say, be as near to the precise position as is necessary for any practical purposes. Then the dis-

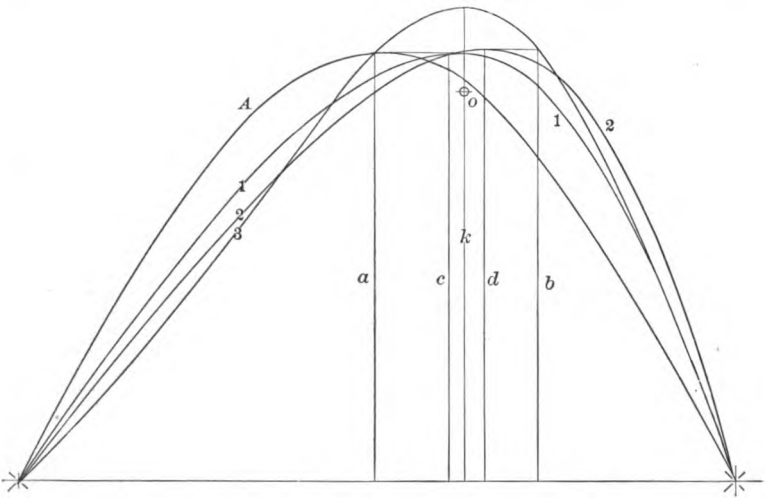


FIG. 70.

tance between *a* and *k* will be a certain fraction of the half-length of the line of abscissas; and if the same fraction of the quadrant be set off to the right from the vertical line in Fig. 69, the phase of the movement at which the piston has the maximum velocity will be practically determined.

But the phases of the movement at which the piston speed is equal to the crank speed, can be graphically determined in a very simple manner.

82. Thus in Fig. 71, the crank CA is vertical. AB is the connecting rod, AM the velocity of A, and Am the longitudinal component, to which Bn must be equal. Drawing nN perpendicular to AB, it is evident that the two triangles AmM, BnN, are similar and equal, so that $BN = AM$. And this accounts for the fact that in Fig 70, curve No. 3 cuts curve A at the extremity of ordinate a; the crank being vertical in both cases. In order to avoid confusion, the other phase is shown *below* the horizontal centre line in Fig. 71;

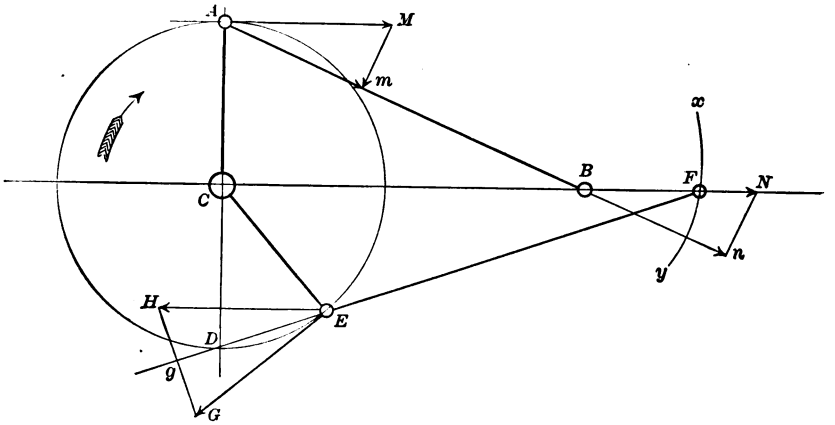


FIG. 71.

the relative positions of the parts being such that the prolongation of the connecting-rod FE shall pass through D, the lowest point of the path of the crank-pin. Let EG, perpendicular to CE, be the velocity of the crank-pin; draw GH perpendicular to FED, also EH parallel to the path of F and consequently perpendicular to CD, then EH will be the velocity of F. But by the construction the triangles CED, GEH, are similar; and since $CE = CD$, it follows that $HE = EG$.

Now, if the lengths of the crank CE and the connecting-rod EF are assigned, there is no geometrical method of

locating E or F. But the locations of both points may be determined with great accuracy, by marking on the edge of a straight slip of paper, two points indicating the length of EF. Then, moving this slip around so that the point E shall always lie on the crank-path, while the edge of the slip passes through D, mark the corresponding location of F for a number of positions of E. The points thus marked determine a curve xy, which cuts the line of travel in the point F; and an arc about this point as centre, with radius equal to the given length of the connecting-rod, intersects the path of the crank-pin at the required point E, determining the corresponding position CE of the crank.

Now, the differences between the results of these four arrangements are very decided, and of no small interest; and it is safe to say that these differences can be more readily compared, and are made more conspicuous, by the diagrams in Fig. 70, than they could be by any other means whatever.

83. As another example, let us consider the two slow advance and quick return motions shown in Figs. 72 and 73. The first is the Whitworth movement, which has already been described; the second was devised by the writer, with a view of avoiding the sliding in the slotted arm of the former. In order to do this, the two cranks are connected by a short link AB; and the method of construction is shown in Fig. 74.

Describe about D a circle with radius DB the length of the driving crank, and divide its circumference so that the arc BMF shall be to the remainder of the circumference as the time of the return is to the time of the advance; draw BF, and bisect it by a perpendicular, which locates the fixed centre C. Next draw the link BA parallel to CD;

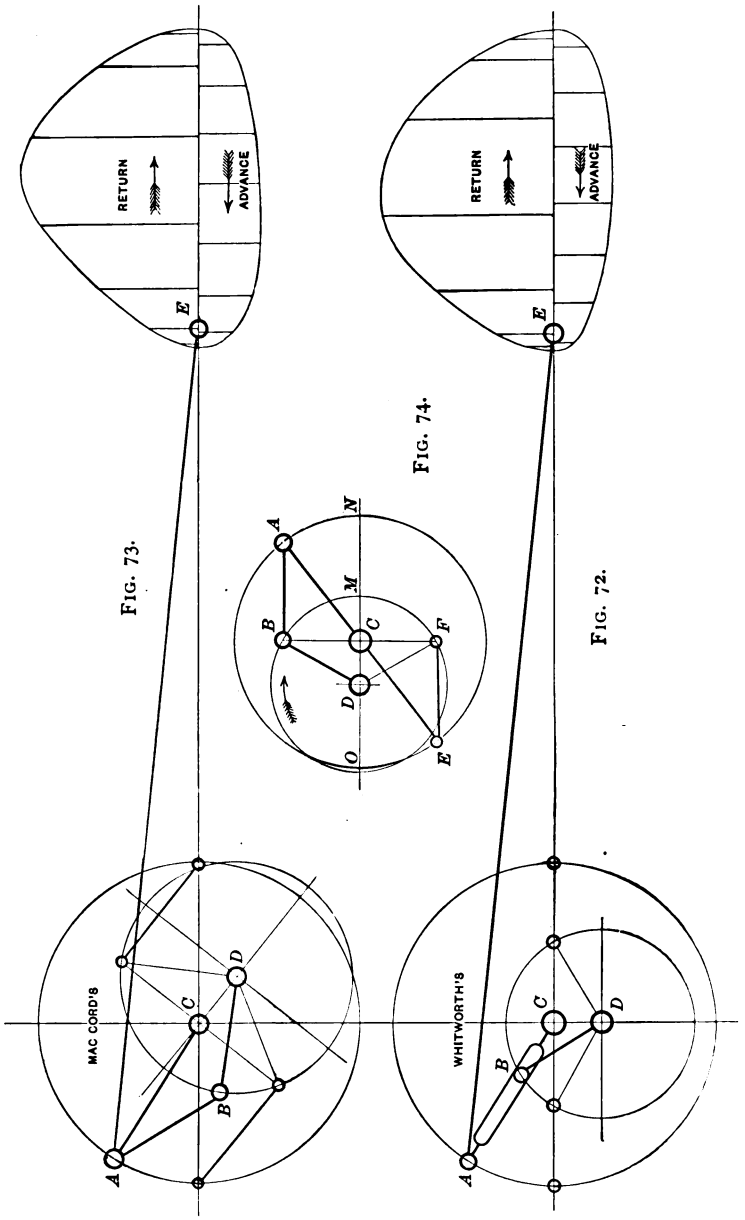


FIG. 73.

FIG. 74.

FIG. 72.

the length of this link may be varied within narrow limits, but must always be greater than *MN* and less than *MO*; and finally draw *AC*, the driven crank. If now we suppose the driver to turn in the direction of the arrow, it will be seen that while *B* describes the arc *BMF*, *A* will have been driven through the arc *ANE*, or 180°, the two cranks then having the positions *DF*, *CE*; and that in order to make *A* describe the other semicircumference and return to its original position, *B* must travel through the remaining portion *FEB* of its own path.

This combination, clearly, is the well-known drag link;

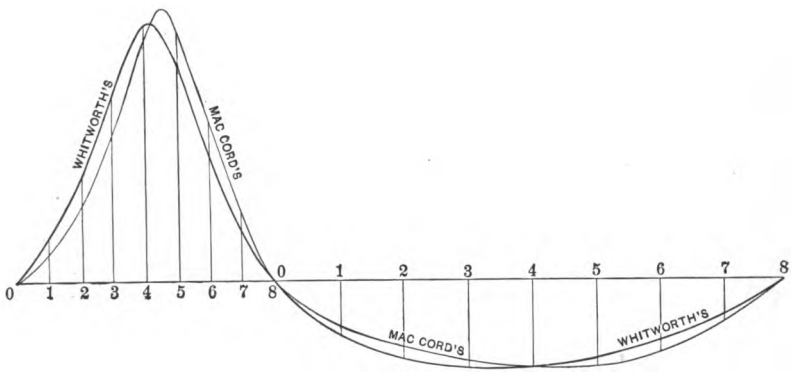


FIG. 75.

but we have never met with any suggestion of using it for this particular purpose. In so applying it, the diameter *ACE* must, evidently, be the line of travel of the remote extremity of the link which moves the tool-holder, as shown in Fig. 73.

Now, these two movements have been drawn on the same scale, and constructed to have the same proportion between the times of advance and return; and there is a pronounced difference between the velocity-space diagrams.

But the most interesting and valuable comparison is that between the velocity-time diagrams, which, drawn on a larger scale, are in Fig. 75 shown as superposed; the line of abscissas, and the scale of the ordinates, being the same for each, exactly as in Fig. 70.

84. Again, comparisons between angular velocities may be advantageously made in the same manner. When one rotating piece has a *constant* angular velocity, this fact has already been illustrated (see Figs. 47, 58, 61). But in Fig. 76, the two equal reciprocating levers, BC and EF, both move with varying angular velocities; and it may be desirable to compare their variations.

The driving crank AD actuates the lever BC by means

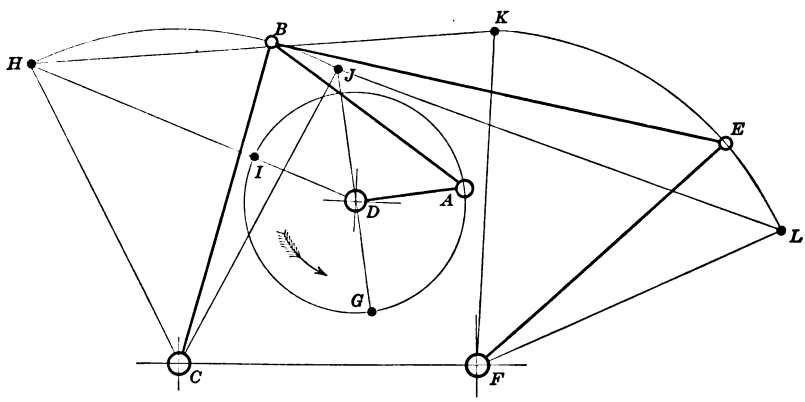


FIG. 76.

of the link AB, thus causing B to traverse the arc JH, exactly as in Figs. 51 and 63; and BC actuates EF by means of another link BE. These two levers were made of the same length, it may be stated, simply to lessen the labor of construction; for taking that length as unity, the linear velocities of B and E at once represent their angular veloci-

ties. And the values of these velocities are the ordinates of the two curves shown in Fig. 77, the one drawn in the heavy line relating to the lever BC, and the other to the lever EF; the area of the latter, it is seen, is the greater, as it obviously should be, since the arc KL is greater than HJ.

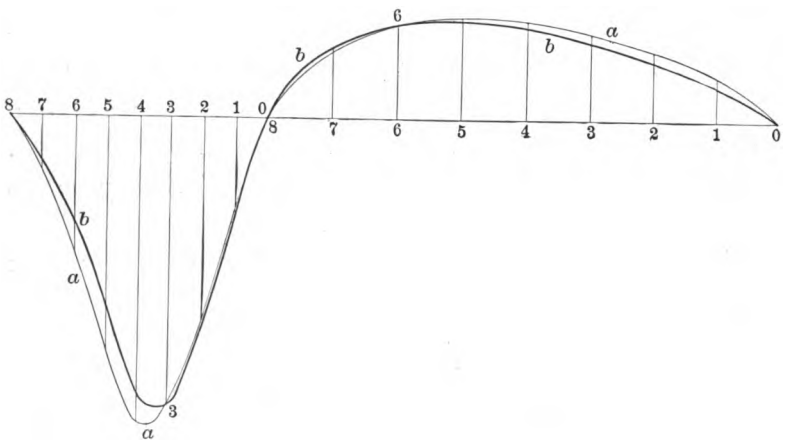


FIG. 77.

85. Now, if we divide the ordinates of one curve by those of the other in their order, we obtain a series of fractions; each of these has a numerical value, which is either equal to, or greater or less than, unity, and expresses the value of the velocity ratio at the corresponding instant. And by setting up ordinates with these values, we may construct a curve exhibiting the variations of the *angular velocity ratio*. By inspection of Fig. 77, it will be seen that the value of this ratio will be unity where the curves intersect, as at *6* on the right and at *3* on the left, since there the ordinate is the same for each.

And since these ordinates become more nearly equal as we approach the zero points, it is clear that unity is the

limiting value when the ratio becomes $\frac{0}{0}$, as it does when the driving crank in Fig. 76 is on the dead centres.

In the nature of things a ratio conveys no idea of direction, so that in constructing such a curve the ordinates may all be set up on the same side of the line of abscissas, thus requiring but one line for unity value; and this has been done in Fig. 78, in which the line of abscissas is equal in

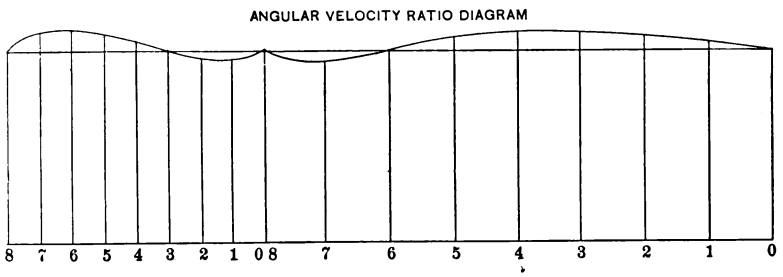


FIG. 78.

length to that in Fig. 77, and subdivided in the same manner.

The values of the ordinates were obtained by dividing those of the curve *aaa* in Fig. 77, by those of the curve *bbb*; but it is proper to state that all this work has been done upon a small scale, and for illustrative purposes only, and of course the results are by no means to be regarded as equal in accuracy to those exhibited in many of the preceding diagrams

This then shows that variable velocity ratios *may* be graphically represented by a curve. But a ratio is essentially a compound idea; and though its value at any instant may be represented by the ordinate of a curve, as in Fig. 78, such a curve conveys no idea of the actual velocities which

determine that value. Of the two, then, Fig. 77 is far more explanatory and withal more readily understood, since both terms of the fraction are shown at the same time and on the same ordinate. Still, there may be cases in which a comparison of variations in angular velocity ratio might be desirable; and then the construction shown in Fig. 78 may prove to be of great utility.

ACCELERATION.

86. We pass now to the consideration of some graphical operations which, while not directly connected with the construction of velocity diagrams as heretofore considered, are nevertheless closely related to it, and are of at least equal interest.

In Fig. 79, mark on the line of abscissas, beginning at zero on the left, the equidistant points *1*, *2*, *3*, etc., each representing an instant; then the spaces between them will represent equal periods of time. Set up at *1* an ordinate *1a*, representing on any convenient scale the distance traversed by a moving point during the first period. Let the distance traversed during the first two periods be represented by an ordinate *2b*, twice as great as *1a*. At the end of three periods, let the distance be three times as great, at the end of four, four times as great, as indicated by the ordinates *3c*, *4d*, and so on. It will then be obvious that the line drawn through the points *a*, *b*, *c*, *d*, etc., will be a right line passing through the zero point; and also, that the difference between any two ordinates will represent the distance traversed by the moving point during the intervening period —thus, $4d - 1a = hd$, the distance traversed during three of the equal periods first set off. Similarly, drawing through

c the horizontal line ce, we have de as the space traversed in the time ce, and since $\frac{\text{space}}{\text{time}} = \text{velocity}$, $\frac{de}{ce}$ represents the velocity at the instant 3; and if we regard ce as a unit of time, de itself will be equal to the velocity; and accordingly $3d'$ is set up equal to de. In this case the velocity is clearly uniform—and may accordingly be represented by a horizontal line through d'; as it is quite obvious that for any

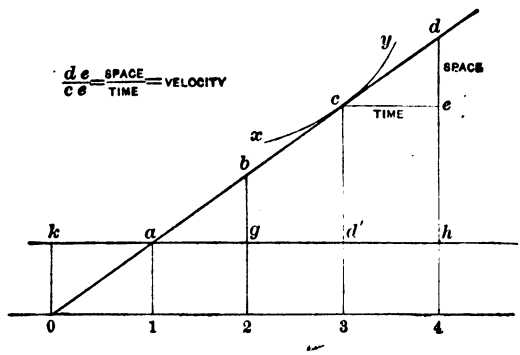


FIG. 79.

other instant, as for instance 1, the same process will give for the velocity, $\frac{bg}{ag} = \frac{de}{ce}$.

The line drawn through the extremities of the ordinates (in this case ad) may properly be called the *line of displacement;* and it is straight because the differences between successive equidistant ordinates are equal. Had they been unequal, that line would have been a curve; as for example xy; and let us suppose the conditions to be such that xy is tangent to ad at the point c. The velocity of the moving point at the instant 3, being measured by the distance which it would travel in a unit of time if that velocity were uniform,

would clearly be the same as that just determined. And had that curve of displacement been assigned, the velocity at that instant would be ascertained by erecting an ordinate cutting the curve at *c*, drawing through that point a tangent line and also a horizontal line of a length *ce* representing the unit of time, and erecting the vertical line *ed*.

87. To illustrate a practical application of this process, let us take the crank and connecting-rod movement of Fig. 80; supposing the rotation to be uniform, the circumference is divided into equal parts, and the corresponding positions of the cross-head pin are determined as usual. Then the distances *01*, *02*, *03*, etc., are set up as ordinates at the equidistant points *1*, *2*, *3*, etc., in Fig. 81, the spaces between them representing equal times; and thus we construct the time displacement curve as shown. The velocity at *4* is determined as above explained, and the same process being repeated for the other instants, we have a velocity diagram, derived from the curve of displacement; the whole process being entirely different from anything previously described. And setting up the ordinates of this curve at the correspondingly numbered positions in Fig. 80, we can construct the velocity-space diagram.

Now in order to test the accuracy of this new method by comparing it with the old, we must know the circumferential velocity of the crank-pin; this is found by means of the fact that, as has previously been shown, it is equal to that of the cross-head pin when the crank is vertical, that is to say, to the ordinate *3-3*. Applying this test, the results of the two methods were found to be identical, in this case; but so happy a coincidence is not always to be looked for, since the new process depends upon the accurate drawing of a series of tangents, which in dealing with a curve of

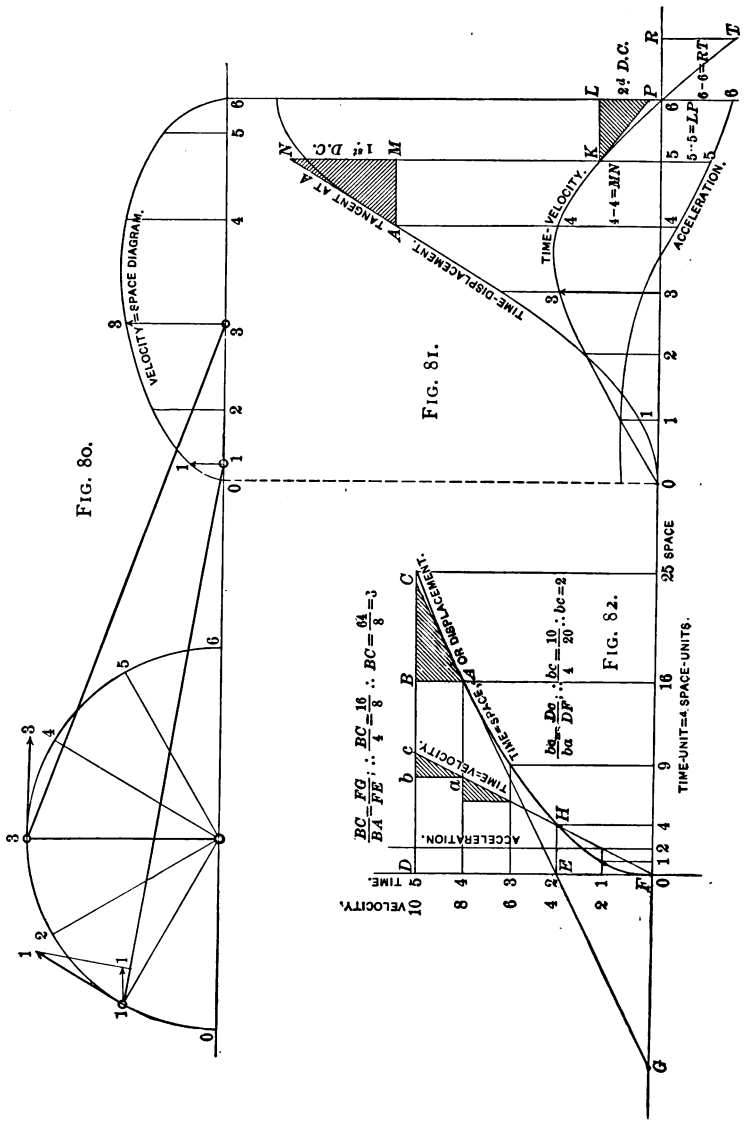


FIG. 80.

FIG. 81.

FIG. 82.

unknown geometrical properties is sometimes quite a difficult, not to say an uncertain task; particularly if the curve is very flat.

88. As to the theoretical correctness of this process, however, the argument given in connection with Fig. 79 would appear conclusive; but corroborative evidence is found by its application to a displacement curve of known properties, to which the tangent can be drawn by geometrical construction; of which Fig. 82 is a good example. In order to give the diagram a more convenient form, the times are set up on the vertical line, as at *1*, *2*, *3*, etc., and the corresponding spaces, as *1*, *4*, *9*, etc., are laid off on the horizontal line, and the time-unit is four times the space-unit. Since the spaces are proportional to the squares of the times, the displacement curve is a parabola, of which *F* is the vertex and the horizontal line through *F* is the axis; the tangent to which at any point is most conveniently drawn by aid of the property that the subtangent is bisected at the vertex. To draw the tangent at *A*, then, set off $FG = F16$, and draw *GA*. By the same process as in Fig. 81, then, we have $\frac{BC}{BA}$ as the value of the velocity at the instant *4*, and *BA* being unity, we set off $4a = BC$.

Now to find the numerical value of *BC*, we have by similar triangles, $\frac{BC}{BA} = \frac{FG}{FE}$, or $\frac{BC}{4} = \frac{16}{8}$, whence $BC = 8$. If this process be repeated, we shall find in a similar manner that the velocities at the instants *1*, *2*, *3*, etc., are respectively *2*, *4*, *6*, etc., showing that the velocity diagram is a straight line, and agreeing with the well known law of falling bodies, that the velocity varies directly with the time, and the space with the square of the time.

89. Now, just as the difference between two consecutive ordinates of the displacement curve indicates the velocity (or the rate at which the displacement is changing), so the difference between two consecutive ordinates of the velocity curve indicates the acceleration, or rate at which the velocity is changing. And by applying to the latter curve the same process that was applied to the former, we may determine an acceleration curve as in Fig. 81, exhibiting this rate of change at each instant of the motion. Thus KP is tangent to the velocity curve at K, through which point the horizontal line KL is drawn, and $\frac{PL}{KL}$ represents the acceleration at the instant 5; since KL = unit time, the ordinate 5–5 is made equal to LP. But it is measured *downward* from the line of abscissas, because P is below the horizontal KL, indicating that the velocity is undergoing retardation; which it obviously must be, since the maximum velocity was reached between the instants 3 and 4.

In Fig. 82, the time-velocity is represented by a right line, and as in Fig. 79, the application of this process will produce another right line, parallel to the axis on which the times are measured. The distance of this line of acceleration may be found thus: By similar triangles we have $\frac{bc}{ab} = \frac{Dc}{DF}$, or $\frac{bc}{4} = \frac{10}{20}$, whence $bc = 2$; which accords with the other law of falling bodies, that the acceleration is constant, and equal to the velocity acquired at the end of the first instant, counting from the state of rest as zero.

90. This process may be and has been described as one of graphical differentiation; and correctly, too; for the ratio $\frac{MN}{AM}$, in Fig. 81, will remain the same, no matter how short

AM may be, and when A and M become consecutive, the fraction $\frac{MN}{AM}$ will become, in the language of the calculus, merely $\frac{dy}{dx}$, or the first differential coefficient of the curve of displacement, from which the velocity curve is derived by differentiation.

And similarly, $\frac{PL}{KL}$, representing the first differential coefficient of the second curve, represents also the second differential coefficient of the original one. But the explanation given in connection with Fig. 79 will, we think, make the matter clear to those not familiar with the language of analysis.

It is to be observed that in Fig. 81, the final ordinate *66* of the acceleration curve is equal to TR, to obtain which a tangent PT is drawn to the curve at its extremity, and $6R$ is set off equal to the unit of time. Also, PT makes a greater angle with the horizontal, than any other tangent which slopes downward toward the right; and of those which slope downward toward the left, the greatest angle is made by the tangent at the zero point. Consequently, the maximum ordinates of acceleration and retardation respectively, correspond to the beginning and the end of the stroke; which is clearly as it should be in this case.

91. But it is by no means true in all cases; a good illustration of this is given in Fig. 83, where the velocity curve is that of the "quick return" of a Whitworth movement. Here it is evident that the inclination of the curve to the horizontal (which is measured by that of its tangent), increases as we recede from the zero point, until the point of contrary flexure is reached, when it reaches its maximum,

and so does the ordinate of the curve of acceleration. When the velocity is at a maximum, it is clear that the acceleration will be nil; the tangent is at that instant hori-

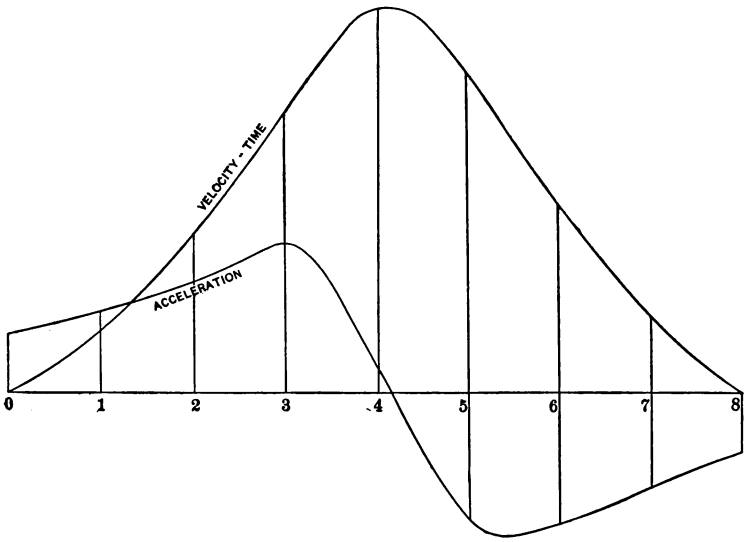


FIG. 83.

zontal, and after that it will slope downward to the right: and similar reasoning to the above applies to the negative acceleration, or retardation.

Since the motion begins at zero and ends at zero, it is clear that the retardation and the acceleration must exactly balance, so that the area of that part of the curve above the line of abscissas should be equal to the area of the part below.

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